CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

C^*-valued G-Contraction and Fixed Points

by

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Dedicated to my dearest father and loving mother who are still alive in my heart

"Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality."

Richard Courant

Abstract

Recently, Ma et al. introduced the notion of C^*-valued metric spaces and extended the Banach contraction principle for self mappings on C^*-valued metric spaces. Motivated by the work of Jachymski, in this thesis we extend and improve some fixed point results on C^*-valued metric space satisfying the contractive condition for those pairs of elements from the metric space which form edges of a graph. Our results generalize and extend the main result of Jachymski and Ma et al. and those contained therein. We also establish some examples to elaborate our new notions and to substantiate our results.

List of Publications

(1) D. Shehwar, T. Kamran, C^*-valued G-Contractions and fixed points, *J. Inequal. Appl.*, 2015:304, (2015).

(2) D. Shehwar, S. Batul, T. Kamran, A. Ghura, Caristi Fixed Point Theorem on C^*-algebra Valued Metric Space, *J. Nonlinear Sci. Appl.* 9, (2016) 584-588.

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Preface

For the last few decades, fixed point theory has turned out to be one of the promptly growing areas of research. Theorems concerning the existence and properties of fixed points are known as fixed point theorems. Fixed point theory is a beautiful mixture of analysis, topology, and geometry which plays crucial role in many branches of pure, applied and commputational mathematics. In 1866, Poincare [77] was the first in this feild. Then Brouwer [23] in 1912, proved a fixed point theorem for the solution of $Tx = x$ which was further extended by Kakutani [52]. In particular fixed point theorems are used to apply the successive approximations to establish the presence and distinctiveness of solution, particularly of differential equations. This method is associated with the eminent mathematicians for example Liouville, Picard, Peano, Fredholm and Cauchy. The existence of a fixed point is therefore of paramount importance in several areas of mathematics and other sciences.

In 1922, the Banach contraction principle (BCP) [11] came into the existence which is considered as one of the fundamental principles in functional analysis. Banach is ascribed with placing the underlying ideas into an abstract framework suitable for proving the existence of solutions for many non-linear problems. Due to the massive applications of Banach contraction principle, it has become the center of focus for many mathematicians. Later on, it was generalized and developed by Kannan [55]. The fixed point theory has been studied and generalized in different spaces and various fixed point theorems are developed. Drastic changes took place with work of Ciric [28] followed by the work of Rhoades [81] and Kirk [60] on non expansive mappings. Also, the work of Park [73] and Sadovski [85] have made valuable contribution by considering new types of mapping conditions.

The Banach contraction principle states that " if T is a self mapping on a complete metric space (X, d) and there exists $\alpha \in (0, 1)$, such that

$$d(Tx, Ty) \leq \alpha d(x, y), \qquad \forall\ x,\ y\ \in X, \tag{1}$$

then, T has a unique fixed point."

Caristi's fixed point theorem [24] is a beautiful extension of Banach contraction principle. According to this result " if X is a complete metric space and $T : X \to X$ is a self map satisfying the following property:

$$d(x, Tx) \leq \phi(x) - \phi(Tx) \qquad \forall\, x \in X, \tag{2}$$

where $\phi : X \to \mathbb{R}^+$ is a lower semi continuous map. Then T has a fixed point." There are many different ways to generalize the Banach contraction principle (BCP). Although enormous developement has been done in the feild of fixed point theory but there are still several intresting interrogations to respond regarding how and to what extent the theory can be developed and comprehended. One of such questions arises when we notice that the Banach contraction principle requires that T satisfies the contractive condition on each point of $X \times X$. So, one possible way to generalize the BCP is to impose a suitable condition on ordered pairs from $X \times X$ in such a way that (1) holds only on a subset of $X \times X$ and the mapping still has a fixed point. Ran and Reurings [79] took the initiative towards this direction. They showed that the mapping T still has a fixed point subject to the completeness of the partially ordered set X provided that T be contractive for those pairs which are related. Afterwards, many authors contributed a lot in the fixed point theory on partially ordered metric spaces, see for example, Bashkar and lakshmikanthm [20], Neito and Roriguez [68], Petrusel and Rus [76], and Neito et al. [71].
Jachymski [46] further extended this idea and replaced the ordered structure with a graph whose vertices coincide with the metric space X. He showed that T has a fixed point if contractive condition holds for those ordered pairs which are the edges of a graph. Subsequently, many authors extended his idea in different ways, see for example ([3], [53], [86], [89],[41], [14], [9]).

On the other hand one may consider the more generalized space X instead of just considering it as a metric space. Several attempts have been made in this direction as well and many interesting generalizations of a metric space were introduced by different authors in order to generalize BCP and Caristi's fixed point theorem. In 1989, Bakhtin [10] introduced the idea of b-metric spaces which he then used to replace the traditional metric space for the generalization of the famous Banach contraction principle. This quest led many mathematicians to improve fixed point theory in b-metric spaces [29]. Several authors undertook further development in this direction and established some exciting results, see an excellent survey by Van An et al. [7]. In the same pursuit, the study of fixed point theory endowed with graph occupied a prominent place in many aspects. Motivated by this idea of proving the fixed point results in the setting of graph, many authors established some important results regarding the existence of fixed point in b-metric spaces equipped with a graph see for example [65, 89].

In 2007, cone metric space was introduced by Huang and Zhang [45]. This is a motivating broad view of a metric space where the metric is replaced by the mapping with values in an ordered Banach space. Huang and Zhang introduced the basic definition and proved the properties of sequence in cone metric spaces. They also obtained several fixed point theorems for contractive single valued maps in such spaces. Cone metric space has become an exciting subject for many authors (for example [27, 37, 62]).

These authors stimulated many mathematician to further generlize this idea. In this pursuit, Ma et al. [63] recently introduced an attention-grabbing generalization of metric space so called C^*-algebra valued metric space or C^*-valued metric space and generalized the BCP in this new setting. The authors introduced the new concept by replacing the ordered Banach space with a C^*-algebra. This is undoubtedly a generalization of cone metric spaces.

Motivated by this work, we bring in light some new concepts of C^*-valued G-contractions, G-lower semi continuity and C^*-valued Caristi type mappings in the setting of C^*-valued metric space. In this thesis, we have established some fixed point results on C^*-valued metric space endowed with a graph. These results extend and generalize the results given by Jachymski [46] and Ma et al. [63].

This thesis is organized as follows. In Chapter **1**, we have given a short note on the history of graph. We have also recollected some basic concepts of graph theory which are useful in understanding and developing the new fixed point results. To establish the new results in C^*-valued metric space, we need a brief introduction of C^*-algebras. Therefore in the second section of this chapter, we introduce the fundamentals of C^*-algebras by providing suitable examples.

Chapter **2** is devoted to brief literature review of metric fixed point theory. Moreover we imitate some necessary and relevant results and provide examples for elaboration. Proofs of the theorems and details of the topics can be found in the books and articles referred for.

In Chapter **3**, we describe the notion of C^*-valued metric space. We also give some important examples to illustrate the idea that it generalizes the concept of a metric space. We also refurnished the fixed point results given by Ma et al. [63].

In the first section of Chapter **4**, we introduce the notion of C^*-valued G-contraction and provide some examples to show the generality of our new notion. In section 2, we obtain a fixed point result for C^*-valued G-contraction which generalizes the results by Ma et al. [63] and Jachymski [46]. The result of this section appeared in

"D. Shehwar, T. Kamran, C^*-valued G-Contractions and fixed points, *J. Inequal. Appl.*, 2015:304, (2015)."

We also define Caristi type contraction in the next section and obtain two fixed point results in the setting of C^*-valued metric space. Theorem 4.3.6 and Theorem 4.3.7 appeared in

"D. Shehwar, S. Batul, T. Kamran, A. Ghura, Caristi Fixed Point Theorem on C^*-algebra Valued Metric Space, *J. Nonlinear Sci. Appl.* 9, (2016) 584-588."

Later, we extend the the theorem in [91] by proving it in the setting of graph.

In the next section of this chapter, by using certain contractive conditions intrduced in [12], [44], we define a new notion of C^*-valued G-contractive type mapping. We establish a fixed point result in the setting of C^*-valued metric space. Lastly, we obtain a result on b-metric space on the same setting and provide examples to prove the generality.

Chapter 1

Basic Definitions

In this chapter, we include definitions and notations to make the communication of profound ideas easy. We recollect some basic results to make the arguments precise and convincing. Although the title of each section is indeed a huge discipline of mathematics, but we strictly confine ourselves to the topics required in the rest of the thesis. Subsequently, let us represent by $\mathbb{R}$ the set of real numbers, $\mathbb{N}$ the set of natural numbers, $\mathbb{R}^+$ the set of positive real numbers and $\emptyset$ the empty set.

1.1 Graph Theory

Before giving the formal definitions from graph theory, we would give a brief overview of the history and development of graph theory which transfigured many complicated problems. Graph theory can be offhandedly defined as the study of graphs and graphs are mathematical structures used to model the real life problems by constructing the pairwise relations between objects from certain areas.

Leonhard Paul Euler (1707-1783) was a Swiss mathematician, who spent most of his life in Russia and Germany. Euler settled a renowned unsolved problem of his time called the **Königberg** bridge problem. The **Königberg** is an old city, which was formerly in Germany. Now-a-days it is located in Russia and called Kalingrad. River Preger flows through the city and divide the city into four land masses. The city has seven bridges connecting those four land areas. People of Königberg always inquired that is there any route around the city which would cross the seven bridges once. Many of them tried this task and concluded that it is impossible.

The picture given below shows the map of Königberg during Euler's time. It is highlighting the seven bridges and the river Preger.

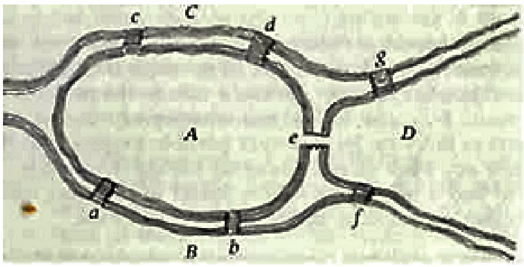

FIGURE 1.1: A map of the city Königberg locating seven bridges

In 1736, Euler discovered the solution of this problem in terms of graph theory. He first proved that it was not possible to walk through the seven bridges by crossing each bridge exactly one time. Euler interconnected the other mathematicians on the problem [94], and wrote an article on this problem. In this article, he abstracted the case of Königberg by eliminating all unnecessary features. He represented the problem by a graph consisting of "dots" that represent the land masses and the line segments representing the bridges that connected those land masses. Hence FIGURE 1.1 can be represented by the following graph:

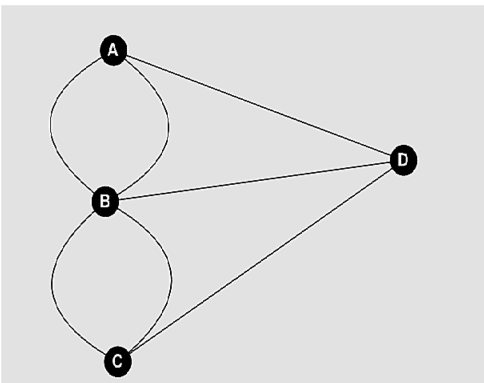

FIGURE 1.2: Graph for the Königberg problem

This graph definitely simplifies the problem. If somebody tries to trace the graph with a pencil without actually lifting it, will realize soon that it is not possible. Euler proved that it is not possible and also explained that why this is not possible to trace the graph without traversing the edges. He gave the concept of degree of nodes/vertices. The degree of node can be defined as the number of edges touching a given node. The solution given by Euler is that any given graph can be traversed with each edge traversed exactly once if and only if it has zero or exactly two vertices with odd degrees. Such graph is called a **Eulerian circuit or path.** He concluded this solution in the following words:

"There must be exactly two vertices at the starting and ending of the trip. If it has even vertices, then we can easily come and leave the vertices without repeating the edge twice or more."

Once the situation of seven bridges of Königberg, was presented in terms of graph, it was simplified as the graph had just four vertices, with each vertex having odd degree. Hence according to Euler's conclusion these bridges cannot be traversed exactly once. Using Eulerian circuit or path, we can solve number of real life situations. If we wish to create a graph for bridges of Königberg, we will create a path to make the degree of two vertices even. The other two vertices which are the starting and ending vertices should be of odd degree. Ultimately the resulting graph is shown below.

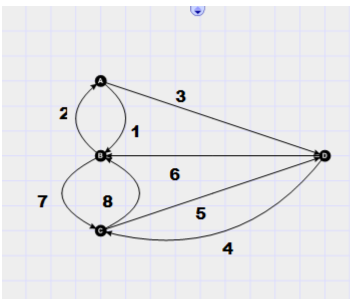

FIGURE 1.3: A graph showing the solution for the Königberg problem

Now, we recall some basic definitions from graph theory which can be found in any standard text such as [49].

Definition 1.1.1.

A ***graph*** is a pair of two sets:

(i) the ***vertex set*** denoted by $V(G)$, is a nonempty set which contains all vertices of the graph,

(ii) the set of ***edges*** denoted by $E(G)$, is a binary operation on $V(G)$.

The most common representation of a graph $G = (V(G), E(G))$ is by means of a diagram, in which vertices are represented by points and edges as line segments joining its vertices. Actually, a graph is a mathematical model which is commonly used to demarcate many real life problems.

Definition 1.1.2.
A ***loop*** is an edge e of G whose end vertices are the same vertex.

Definition 1.1.3.
Let $G = (V(G), E(G))$ be a graph. Then the set

$$\Delta \subset V(G) \times V(G) = \{(x, x) : x \in V(G)\}$$

is called the ***diagonal*** of the graph G.

Example 1.1.4.
For the graph given in the following figure:
$V(H) = \{A, B, C, D, E\}$ and $E(H) = \{a, b, c, d, e, f, g, h\}$.

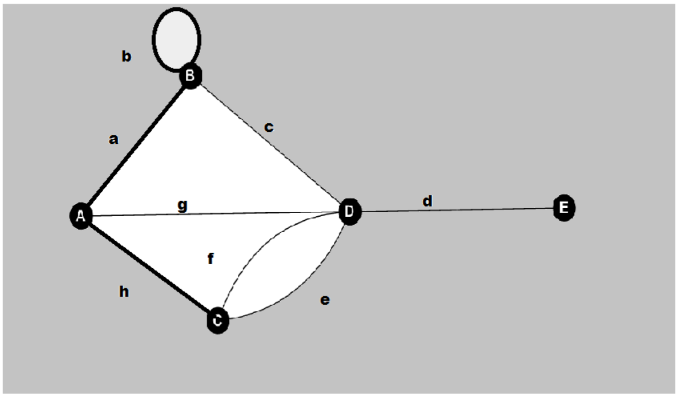

FIGURE 1.4: A graph having multiple edges and loops

Definition 1.1.5.
A graph is *directed* if each edge is specified with the direction from one vertex to the other.

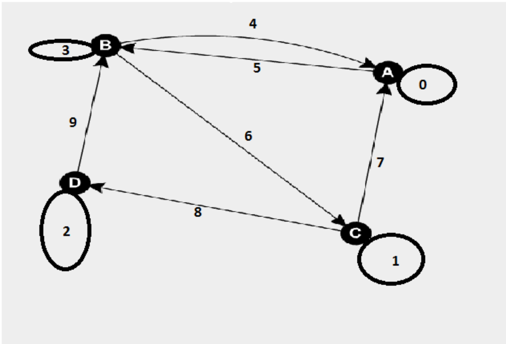

FIGURE 1.5: A directed graph having multiple edges and loops

Definition 1.1.6.

Two edges having the same end points are called ***parallel edges***.

Definition 1.1.7.

A graph is ***simple*** if it has no parallel edges and loops.

Definition 1.1.8.

Let G be a graph. If we change the direction of all edges of G, we will get another graph which is denoted by G^{-1}. Thus we have

$$E(G^{-1}) = \{(a,b) \ \in V(G) \times V(G) : (b,a) \ \in E(G)\}.$$

If we ignore the direction of the edges of G, we will get an undirected graph $\tilde{\mathcal{G}}$. For our convenience we will always consider $\tilde{\mathcal{G}}$ as directed graph and $E(\tilde{\mathcal{G}})$ as symmetric set which implies

$$E(\tilde{\mathcal{G}}) = E(G) \cup E(G^{-1}).$$

Definition 1.1.9.

A graph G is ***symmetric*** if $G = G^{-1}$.

Definition 1.1.10.

Let $G = (V(G), E(G))$ be a graph. Choose a and b from $V(G)$. Then a sequence $\{\ell_i\} : i = 0, 1, 2, \cdots k$ is called a ***path*** in G from a to b of length k if $\ell_0 = a,\ l_i = b$ and $(\ell_{i-1}, \ell_i) \in E(G)$ for $i = 0, 1, 2, \cdots \ell$.

Example 1.1.11.

In FFIGURE 1.6 $\ell_0 = A,\ l_i = K$ and $\{\ell_i\} : i = 0, 1, 2, \cdots k$ ***path*** in G.

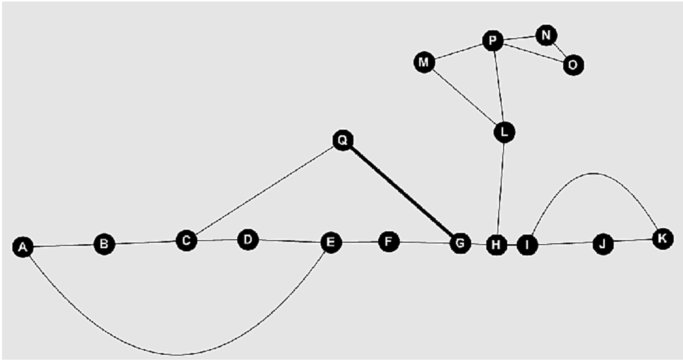

FIGURE 1.6: Path diagram

Definition 1.1.12.
If $G = (V(G), E(G))$ be a graph where $E(G)$ is a symmetric set, then the ***sub-graph*** associated with some point $a \in V(G)$, denoted by G_a, is the part of the graph G that contains all those edges and vertices which are enclosed in some path beginning at a. A sub-graph is also known as a component of G containing a.

Example 1.1.13.
Following diagram contain three connected components.

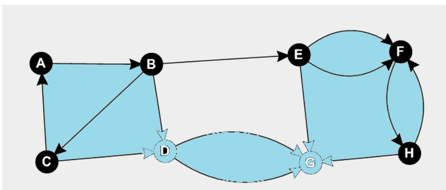

FIGURE 1.7: Three connected components in a graph

1.2 Some Notions from C^*-Algebras

In this section, we summon up some notions from C^*-algebra . For detailed discussion, we refer the readers to [31, 34, 61, 66].

Definition 1.2.1.
An ***algebra*** over $\mathbb{C}$ is a vector space $\mathbb{A}$ with product $(x_1, x_2) \mapsto x_1 x_2$ for each $x_1, x_2 \in \mathbb{A}$ such that

1. $x_1(x_2x_3) = (x_1x_2)x_3 \quad \forall\, x_1, x_2, x_3 \in \mathbb{A}$;
2. $x_1(x_2 + x_3) = x_1x_2 + x_1x_3, \;\; (x_2 + x_3)x_1 = x_2x_1 + x_3x_1$;
3. $(\alpha x_1)(\beta x_2) = (\alpha\beta)x_1x_2$.

Definition 1.2.2.
A $*$***-algebra*** is an algebra over $\mathbb{C}$ if there is an involution map $* : \mathbb{A} \to \mathbb{A}$ satisfying the following conditions:

1. $(x_1 + x_2)^* = x_1^* + x_2^* \quad \forall\, x_1, x_2 \in \mathbb{A}$;
2. $(cx_1)^* = \bar{c}x_1^* \quad \forall\;\, x_1 \in X,\; c \in \mathbb{C}$, where $\bar{c}$ is complex conjugate of c;
3. $(x_1x_2)^* = x_2^*x_1^* \quad \forall\;\, x_1, x_2 \in X$;
4. $(x_1^*)^* = x_1 \quad \forall\, x_1 \;\in X$;

A $*$-algebra with an identity is called a ***unital*** $*$***-algebra***.

Definition 1.2.3.
A ***normed algebra*** is a complex algebra $\mathbb{A}$ with a norm $\|\cdot\| : \mathbb{A} \to \mathbb{R}$ satisfying the following:

$$\|x_1x_2\| \leq \|x_1\|\|x_2\| \quad \forall\, x_1, x_2 \in \mathbb{A}.$$

Definition 1.2.4.
If a normed algebra $\mathbb{A}$ is complete then $\mathbb{A}$ is called a ***Banach algebra***.

Definition 1.2.5.
A Banach algebra $\mathbb{A}$ with involution is called a C^****-algebra*** if it satisfies:

$$\|x^*x\| = \|x\|^2 \quad \forall\, x \in \mathbb{A}. \tag{1.1}$$

Example 1.2.6.
Here are some examples of the C^*-algebra.

(a) Consider the set $C(X) = \{g \mid g : X \to \mathbb{C}\}$, where X is a compact space. Multiplication and scalar multiplication on $C(X)$ are defined as below:

$$g_1.g_2(x) = g_1(x)g_2(x); \qquad (kg)(x) = kg(x).$$

$C(X)$ is a C^*-algebra with norm,

$$\|g\| = \sup_{x \in X} |g(x)|$$

and involution $* : C(X) \to C(X)$ is defined as

$$g^*(x) = \overline{g(x)}.$$

One can very easily verify $C(X)$ as a commutative, unital C^*-algebra, for which e is the identity element defined as

$$e(x) = 1 \ \ \forall \ \ x \in X.$$

(b) Let $\mathbb{A} = \mathbb{M}_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. Identify $\mathbb{M}_n(\mathbb{C})$ with $B(\mathbb{C}^n)$, where the set $B(\mathbb{C}^n)$ contains all linear and bounded operators from n-dimensional Hilbert space $\mathbb{C}^n$ to $\mathbb{C}^n$, with usual matrix operations. Define operator norm i.e,

$$\|x\| = \sup_{\xi \in \mathbb{C}^n} \|x(\xi)\| \ \text{ with } \|\xi\| = 1 \ .$$

If $a = (a_{ij}) \in \mathbb{A}$ we define

$$a^* = (\bar{a_{ji}})\,.$$

Since x and y are bounded operator and $\|\xi\| = 1$

$$\|xy\| = \sup_{\xi \in \mathbb{C}^n} \|xy(\xi)\|$$

$$= \sup_{\xi \in \mathbb{C}^n} \|x(y(\xi))\|$$

$$\leq \|x\| \sup_{\xi \in \mathbb{C}^n} \|y(\xi)\|$$

$$\leq \|x\|\|y\|$$

which implies that $\mathbb{A}$ is a Banach algebra. Also

$$\|x\|^2 = \sup_{\|\xi\|=1} | < x(\xi), x(\xi) > |$$

$$= \sup_{\|\xi\|=1} | < \xi, x^* x\xi > |$$

$$= \|x^* x\|.$$

Hence $\mathbb{A}$ is a C^*-algebra.

Example 1.2.7.

Let $\mathbb{A} = \mathbb{C}^2$ be algebra of all complex pairs. If $w_1, w_2 \in \mathbb{A}$ with $w_1 = (z_1, z_2),\ w_2 = (z_3, z_4)$, then

$$w_1 + w_2 = (z_1 + z_3, z_2 + z_4), \quad kw_1 = (kz_1, kz_2), \quad w_1 w_2 = (z_1 z_3, z_2 z_4).$$

Define $* : \mathbb{A} \longrightarrow \mathbb{A}$ by

$$w_1^* = (\bar{z_1}, \bar{z_2})$$

and $\|\cdot\| : \mathbb{A} \to \mathbb{R}$ by

$$\|w\| = \sqrt{|z_1|^2 + |z_2|^2}.$$

Then one can easily check that

$$\|w_1 w_2\| \leq \|w_1\|\|w_2\|$$

which implies that $\mathbb{A}$ is a Banach algebra. But

$$\|w^* w\| = \sqrt{|z_1|^4 + |z_2|^4} \neq \|w\|^2 = |z_1|^2 + |z_2|^2.$$

Hence $\mathbb{A}$ is not a C^*- algebra.

Definition 1.2.8.

Suppose that X is a real Banach space and $E \subseteq X$. The set E is called a ***cone*** if it satisfies the following conditions,

(i) $E \neq \{0\}$ is closed and non empty.

(ii) $ax_1 + bx_2 \in E$ for all $x_1, x_2 \in E$ and $a, b \in \mathbb{R}$ where $a, b \geq 0$.

(iii) If $x \in E$ and $-x \in E$ then $x = 0$. i. e.,

$$E + (-E) = \{0\}.$$

Definition 1.2.9.
Let $\mathbb{A}$ be a C^*-algebra and $x \in \mathbb{A}$, then the set

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda I - x \text{ is not invertible}\}$$

is called the ***spectrum of an element*** $x \in \mathbb{A}$.

Example 1.2.10.
Let $\mathbb{A}$ be the algebra of all $n \times n$ triangular matrices. If $a \in \mathbb{A}$, say

$$a = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ 0 & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots 0 & \lambda_{nn} \end{pmatrix},$$

it is easy to check that

$$\sigma(a) = \{\lambda_{11}, \lambda_{22}, \cdots \lambda_{nn}\}.$$

Similarly, if $\mathbb{A} = \mathbb{M}_n(\mathbb{C})$ and $a \in \mathbb{A}$, then $\sigma(a)$ is set of all eigenvalues of a.

Definition 1.2.11.
An element $x \in \mathbb{A}$ is called ***positive element*** of a C^*-algebra $\mathbb{A}$ if x is self adjoint i.e., $x = x^*$ and $\sigma(x) \subset [0, \infty)$. The set $\mathbb{A}_+$ denote the set of all positive elements in $\mathbb{A}$.
We say that

$$x \succeq y \text{ if and only if } x - y \in \mathbb{A}_+. \tag{1.2}$$

Example 1.2.12.
Let $\mathbb{A} = \mathbb{C}^2$ be algebra of all complex pairs. Then the of all positive elements of $\mathbb{A}$ is given as:

$$\mathbb{A}_+ = \{(z_1, z_2) : z_1, z_2 \in \mathbb{R} \text{ and } z_1, z_2 \succeq 0\}.$$

Theorem 1.2.13. [66]
"Let $\mathbb{A}$ be a unital C^*-algebra. For $a \in \mathbb{A}_+$ and $\lambda \geq 0$ then

(i) $\lambda a \in \mathbb{A}_+$,

(ii) $a, b \ \in \mathbb{A}_+$ implies $a + b \ \in \mathbb{A}_+$,

(iii) $a, -a \ \ \in \mathbb{A}_+$ implies $a = 0$,

(iv) $\mathbb{A}_+$ is closed.

In other words $\mathbb{A}_+$ is a closed pointed cone."

Theorem 1.2.14. [66]
"Let $\mathbb{A}$ be a C^*-algebra and $x \in \mathbb{A}_+$, then

(i) There exist a unique $y \in \mathbb{A}_+$ such that $y^2 = x$.

(ii) $\mathbb{A}_+ = \{x^*x : x \in X.\}$

(iii) If x and y are self conjugate elements of $\mathbb{A}$ then

$$\theta \preceq x \preceq y \ \text{ then } \ \|x\| \preceq \|y\|,$$

where θ is the zero element of the C^*-algebra $\mathbb{A}$"

Lemma 1.2.15. [66]
According to Murphy, "for a unital C^*-algebra $\mathbb{A}$ with I as an identity

(i) If $x \in \mathbb{A}_+$ with $\|x\| < \frac{1}{2}$, then $I - x$ is invertible and

$$\|x(I - x)^{-1}\| < 1;$$

(ii) Suppose that $x, y \in \mathbb{A}$ with $x, y \succeq \theta$ and $xy = yx$, then $xy \succeq \theta$."

Remark 1.2.16. [66]
"Consider a unital C^*-algebra $\mathbb{A}$. For any $a \in \mathbb{A}_+$ we have

$$a \preceq I \Leftrightarrow \|a\| \leq 1."$$

Definition 1.2.17.
Let $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ be an extended real real-valued function. We say that f is ***upper semi-continuous*** at x_0 if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow \limsup_{x \to x_0} f(x) \leq f(x_0),$$

where lim sup is the limit superior of the function f at x_0. If a function f is upper semi continuous at each point of its domain, then it is called upper semi-continuous function.

Definition 1.2.18.
A real function f is ***lower semi-continuous*** at x_0 if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow \liminf_{x \to x_0} f(x) \geq f(x_0),$$

where lim inf is the limit inferior of the function$f$ at x_0. If a function f is lower semi continuous at each point of its domain then it is called lower semi-continuous function. Following diagrams illustrates the concept more perceptibly.

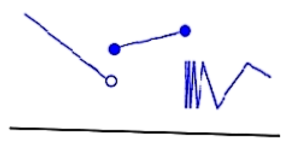

FIGURE 1.8: Upper semi continuity

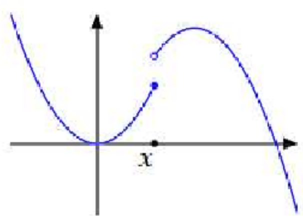

FIGURE 1.9: Lower semi continuity

Example 1.2.19. The function $f : \mathbb{R} \to \mathbb{R}$ defined as:

$$f(x) = \begin{cases} x^2 - 1 & \text{if} \quad -1 \leq x < 0 \\ 2x & \text{if} \quad 0 < x < 1 \\ 1 & \text{if} \quad x = 1 \\ -2x + 4 & \text{if} \quad 1 < x < 2 \\ 0 & \text{if} \quad 2 < x < 3 \end{cases}$$

is lower semi continuous at $x = 1$.

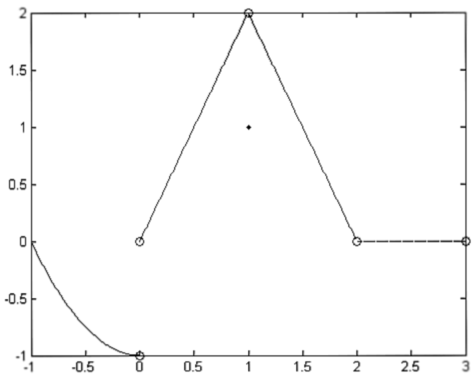

FIGURE 1.10: Lower semi continuity

Chapter 2

Fixed Points of Contractions

The field of fixed points of mappings with adequate contractive conditions has been the focal point of vigorous research activity. It has a extensive applications in different areas such as nonlinear and adaptive control systems, parameterize estimation problems, fractal image decoding and convergence of recurrent networks. The aim of this chapter is to present a brief literature review of metric fixed point theory relevant to our work.

2.1 Contraction Mappings

The fixed point theory is mainly concerned with obtaining conditions on the structure that the underlying space must be endowed with, and the properties of self mapping T on X in order to obtain the existence of fixed points. In this section, we initially discuss some contractive conditions useful for our next discussion.

Definition 2.1.1.
A ***fixed point*** of a mapping $T : X \to X$ of a set X into itself is an $a \in X$ which is mapped into itself that is,

$$Ta = a.$$

Let us use the notation $\mathcal{F}\text{ix}(T)$ for the set of all fixed points of a mapping T , i.e,

$$\mathcal{F}\text{ix}(T) = \{a \in X : Ta = a\}.$$

Example 2.1.2.

Following examples illustrate the idea of fixed point:

1. There are no fixed points for a translation map.
2. A rotation has a single fixed point which is the center of rotation.
3. The mapping $x \mapsto x^2$ has two fixed points that is, $\mathcal{F}\mathrm{ix}(T) = \{0, 1\}$
4. If $Tx = x$ then $\mathcal{F}\mathrm{ix}(T) = \mathbb{R}$.
5. Following diagrams imitate the idea of fixed point beautifully:

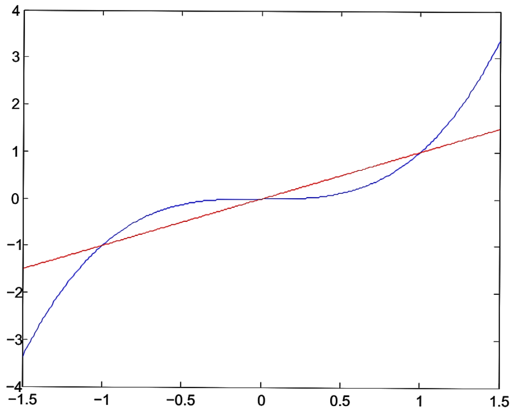

FIGURE 2.1: Fixed points of a function.

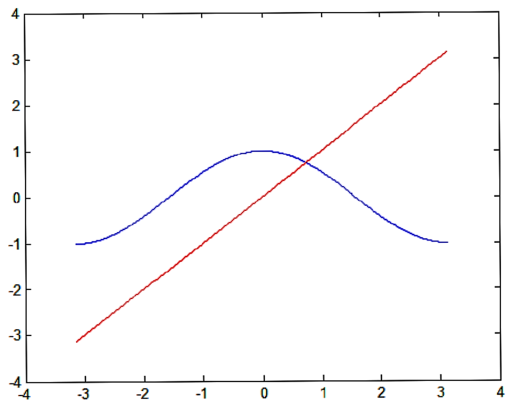

FIGURE 2.2: Unique fixed point

Definition 2.1.3.

(i) Consider a metric space (X, d). A self map $T : X \to X$ is called a " Lipschtizian map", if for a small number $\varrho \geq 0$ the mapping T satisfies the following
$$d(Ta_1, Ta_2) \leq \varrho d(a_1, a_2) \qquad \text{for all } a_1, a_1 \in X. \tag{2.1}$$
The real number ϱ for which (2.1) is true is known as "Lipschtizian constant" for T. It can be easily verified that Lipschtizian map is necessarily continuous.

(ii) Consider a space X induced by a metric d. A self map $T : X \to X$ is called "contractive mapping" if
$$d(Ta_1, Ta_2) < d(a_1, a_2) \qquad \text{for all } a_1, a_1 \in X \text{ with } \ a_1 \neq a_2. \tag{2.2}$$

(iii) A self map T on X is called "nonexpansive" if
$$d(Ta_1, Ta_2) \leq d(a_1, a_2) \qquad \text{for all } a_1, a_2 \in X. \tag{2.3}$$

(iv) Consider a metric space (X, d) . A mapping $T : X \to X$ is called a "contraction" if there exists a small number $\varrho \in [0, 1)$ such that
$$d(Ta_1, Ta_2) \leq \varrho d(a_1, a_2) \qquad \text{for all } a_1, a_2 \in X. \tag{2.4}$$
Geometrically this means that any points a_1 and a_2 have images that are closer together than those points a_1 and a_2. Indeed , the value
$$\frac{d(Ta_1, Ta_2)}{d(a_1, a_2)}$$
does not become greater than the constant ϱ which is strictly less than 1.

Remark 2.1.4.
One can easily observe that every contraction map is contractive and every contractive map is nonexpansive. Hence all of these are Lipschtzian.

The Banach contraction principle is a remarkable contribution towards fixed point theory.

Theorem 2.1.5. (Banach Contraction Principle)

"Let (X, d) be a complete metric space with $X \neq \emptyset$. Assume that $T : X \to X$ is a contraction mapping with contraction constant $k \in (0, 1)$. Then T has a unique fixed point $x \in X$."

The importance of the principle lies in the fact that it not only guarantees existence of a fixed point but it also provides the error bounds. Under the conditions of Banach contraction principle the iterative sequence

$$x_0, \quad x_1 = Tx_0, \quad x_2 = Tx_1, \ \cdots \ x_n = Tx_0, \ \cdots . \tag{2.5}$$

with arbitrary $x_0 \in X$, converges to a unique fixed point x of T. Following are two error estimates:

(i) *priori estimate*

$$d(x_m, x) \leq \frac{k^m}{1-k} d(x_0, x_1) \qquad m = 0, 1, 2, \cdots$$

(ii) and the *posteriori estimate*

$$d(x_m, x) \leq \frac{k}{1-k} d(x_{m-1}, x_m) \qquad m = 0, 1, 2, \cdots$$

2.2 Weak Contractions

The Banach contraction principle has valuable applications using iterative methods for solving linear algebraic equations. It yields sufficient conditions and error bounds. It is also used to show the existence of solution for differential equations. A number of mathematicians have established a variety of generalizations of BCP. In this quest, various interesting results regarding the existence of fixed points have been obtained by changing the contractive conditions. These contractive conditions are definitely weaker than one in (2.1.5) and contain not only $d(a_1, a_2)$ on right hand side of (2.4) but also replacement of a_1 and a_2 in different ways under the mapping T i.e.,

$$d(a_1, Ta_1), \ d(a_2, Ta_2), \ d(a_1, Ta_2), \ d(a_2, Ta_1).$$

Numerous generalizations have been done by considering the above replacements and by imposing some necessary conditions to the structure of a metric space or the mapping. In this section, we discuss some of such conditions with illustrative examples to show the relation between these conditions. We will also provide some fixed point results using these conditions.

Definition 2.2.1. [2]
A self map $T : X \to X$ is said to be weakly contractive if

$$d(Ta_1, Ta_2) \leq d(a_1, a_2) - \psi(d(a_1, a_2)) \qquad a_1, a_2 \in X, \tag{2.6}$$

where ψ is continuous, nondecreasing self mapping on $[0, \infty)$. The function ψ has positive images on $(0, \infty)$, with $\psi(0) = 0$ and

$$\lim_{x \to \infty} \psi(x) = \infty$$

It is clear that weakly contractive map is continuous and includes contraction map as a special case (for choice $\psi(x) = (1 - k)x$).

Example 2.2.2. [97]
Let $X = [0, \infty)$ be equipped with the usual metric $d(x, y) = |x - y|$. Consider self map $T : X \to X$ defined as:

$$T(x) = \frac{2}{3}x + 3$$

and $\psi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \frac{1}{3}t.$$

Then one can easily check that T is weakly contractive.

Definition 2.2.3. [55]
Kannan declares that "a mapping $T : X \to X$ is said to be Kannan's mapping if there exist a $k \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq k\,[d(x, Tx) + d(y, Ty)] \qquad \forall\, x, y \in X. \tag{2.7}$$

Contrary to contraction, contrative, nonexpensive and weakly contractive mappings Kannan's mappings are not essentially continuous."

Example 2.2.4. [97]
Let $\Omega = [0,1]$. A map $T : \Omega \to \Omega$ defined by

$$T(\omega) = \begin{cases} 1-\omega & \text{if} \quad \omega \in [0,1] \quad \text{and } \omega \text{ is irrational}, \\ \dfrac{1+\omega}{3} & \text{if} \quad \omega \in [0,1] \quad \text{and } \omega \text{ is rational}, \end{cases}$$

is a Kannan map on the unit interval and it is continuous (nonexpansive) only at its fixed point $\omega_0 = \frac{1}{2}$.

Example 2.2.5. [74]
Let $\Omega = [0,1]$ equipped with the usual metric d and

$$T\omega = \frac{3}{4}\omega, \quad \forall\, \omega.$$

Then T is a Banach contraction but does not satisfy Kannan's condition. For example if $\omega_1 = 0,\ \omega_2 = 1$, then

$$d(T\omega_1, T\omega_2) = \frac{3}{4}$$

and

$$d(\omega_1, T\omega_1) + d(\omega_2, T\omega_2) = \frac{1}{4}$$

and hence

$$d(T\omega_1, T\omega_2) > d(\omega_1, T\omega_1) + d(\omega_2, T\omega_2).$$

Example 2.2.6. [74]
Let $\Omega = [0,4]$ equipped with the usual metric d and $T : \Omega \to \Omega$ be defined by

$$\begin{aligned} T\omega &= 1, && \text{if } 0 \leq \omega \leq 3 \\ T\omega &= 0, && \text{if } 3 \leq \omega \leq 4, \end{aligned}$$

then,

$$d(T\omega_1, T\omega_2) < \frac{1}{3}\,[d(\omega_1, T\omega_1) + d(\omega_2, T\omega_2)] \ \ \forall\, \omega_1, \omega_2 \ \in \Omega$$

Therefore T satisfies the Kannan's contractive condition with $k = \dfrac{1}{3}$, but T is not a Banach contraction. Hence we can conclude that Kannan's contractive condition and Banach contraction are independent of each other.

Now we state some condition proposed by Chatterjea [26], Ciric , Reich and Rus [28, 82, 84], and Hardy and Rogers [42] respectively which are followed by certain fixed point results:

(C1) "For all $x, y \in X$ and $k \in [0, \frac{1}{2})$ we have

$$d(Tx, Ty) \leq k\left[d(x, Ty) + d(y, Tx)\right]." \tag{2.8}$$

(C2) "For all $x, y \in X$

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty), \tag{2.9}$$

where $a + b + c < 1$."

(C3) "For all $x, y \in X$

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(x, Ty) + fd(y, Tx), \tag{2.10}$$

where $a + b + c + d + e + f < 1$."

Example 2.2.7. [81]
Let $X = [0, 1]$ and $T : X \to X$ be a self map defined by

$$T(x) = \frac{x}{3} \ \text{ for } \ 0 \leq x < 1 \ \text{ and } \ T(1) = \frac{1}{6}.$$

T does not satisfy Banach condition because it is not continuous at $x = 1$. Kannan's condition also cannot be satisfied because

$$d(T(0),\ T(\frac{1}{3})) = \frac{1}{2}\left(d(0,\ T(0)) + d(\frac{1}{3},\ T(\frac{1}{3}))\right),$$

but it satisfies (C2), if we put $a = d = \frac{1}{6}$, $b = \dfrac{1}{9}, c = \dfrac{1}{3}$.

Example 2.2.8. [74]
Let $X = [0, 1]$ and $T : X \to X$ be a self map defined by

$$T(y) = \frac{y}{3} \ \text{ for } \ 0 \leq y < 1 \ \text{ and } \ T(1) = \frac{1}{7}.$$

Then,

$$d(Ty_1, Ty_2) < \frac{1}{3}\left[d(y_1, Ty_2) + d(y_2, Ty_1)\right] \ \ \forall\, y_1, y_2 \in X.$$

Therefore T satisfies (C1) with $k = \frac{1}{3}$, but T does not satisfy the Kannan's contractive condition. For example if $y_1 = 0,\ y_2 = \frac{1}{3}$, then

$$d(Ty_1, Ty_2) = \frac{1}{9},\ d(y_1, Ty_1) + d(y_2, Ty_2) = \frac{2}{9}$$

and hence

$$d(Ty_1, Ty_2) > k\left(d(y_1, Ty_1) + d(y_2, Ty_2)\right) \quad \text{for } k \in \left[0, \frac{1}{2}\right).$$

Example 2.2.9. [74]
Let $X = [0, 4]$ equipped with the Euclidean metric d. Let us define a map $T : X \to X$ by

$$T(x) = \begin{cases} 1 & \text{if} \quad 0 \le x \le 3 \\ 0 & \text{if} \quad 3 \le x \le 4. \end{cases}$$

Then,

$$d(Tx_1, Tx_2) < \frac{1}{3}\left[d(x_1, Tx_1) + d(x_2, Tx_2)\right] \ \ \forall\, x_1, x_2 \ \in X.$$

Therefore T satisfies the Kannan's contractive condition, but does not satisfy the Chatterjea contractive condition because when we take $x_1 = 0,\ x_2 = 3$, then

$$d(Tx_1, Tx_2) = 1,\ d(x_1, Tx_2) + d(x_2, Tx_1) = 2$$

and hence

$$d(Tx_1, Tx_2) > kd(x_1, Tx_2) + d(x_2, Tx_1)$$

for each $k \in [0, \frac{1}{2})$.

Hence from the above two examples, we can conclude that (C2) and (2.7) are independent of each other.
Following are some fixed point theorems using the above weakly contractive conditions

(1) Rhoades [81], Theorem 1 states
"For a complete metric space (X, d), a weakly contractive self map $T : X \to X$ has a unique fixed point."

(2) Nemytzki-edelstein [35, 67] states
" A weakly contractive mapping $T : X \to X$ has a unique fixed point if (X, d) is a compact metric space."

(3) Kannan [55] states
"A Kannan mapping $T : X \to X$ has a unique fixed point if (X, d) is a complete metric space ."

(4) Chatterjea [26] states
"For a complete metric space (X, d), the mapping $T : X \to X$ which satisfies (C1) has a unique fixed point."

(5) Hardy and Rogers [42] states
"For a complete metric space (X, d), the mapping $T : X \to X$ which satisfies (C3) has a unique fixed point. "

Moreover, Branciari [22] generalized the Banach contraction principle. He used contractive condition of integral type and proved the existence of unique fixed point for a self map on complete metric space. Following his foot steps, many authors carried out investigations with this type of contractive condition (see, e.g., [5, 33, 81, 93, 96])

Theorem 2.2.10. (Branciari [22])
"If (X, d) is a complete metric space, $k \in (0, 1)$ and let $T : X \to X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(Tx,Ty)} \psi(s)ds \leq k \int_0^{d(x,y)} \psi(s)ds, \tag{2.11}$$

where $\psi : [0, \infty) \to [0, \infty)$ is Lebesgue integrable, summable on each compact subset of $[0, \infty)$, non negative and for each $\epsilon > 0$, $\int_0^\epsilon \psi(s) > 0$. Then T has a unique fixed point $\xi \in X$ such that for each

$$x \in X, \lim_{n\to\infty} T^n x = \xi."$$

Example 2.2.11. [22]
Let $X = \left\{ \frac{1}{p} : p \in \mathbb{N} \right\} \cup \{0\}$ with Euclidean metric defined as $d(x, y) = |x - y|$. Since X is closed subset of $\mathbb{R}$, therefore (X, d) is a complete metric space. Consider

a map $T : X \to X$ defined by

$$T(x) = \begin{cases} \dfrac{1}{p+1} & \text{if } x = \dfrac{1}{p},\ p \in \mathbb{N}, \\ 0 & \text{if } x = 0 \end{cases}$$

then it satisfies (2.11) with

$$\psi(s) = s^{\left(\frac{1}{s}-2\right)}[1 - \log s] \ \text{ for } \ s > 0,\ \psi(0) = 0 \ \text{ and } \ k = \frac{1}{2}.$$

In this context one has

$$\int_0^\tau \psi(s)ds = \tau^{\dfrac{1}{\tau}},$$

so that (2.11), for $x \neq y$, is equivalent to

$$d(Tx, Ty)^{\dfrac{1}{d(Tx,Ty)}} \le kd(x, y)^{\dfrac{1}{d(x,y)}}. \tag{2.12}$$

The next step is to show the existence of (2.12). Let $p, q \in \mathbb{N}$ with $q > p$, $x = \dfrac{1}{p}$, $y = \dfrac{1}{q}$, then we have

$$d(Tx, Ty)^{\dfrac{1}{d(Tx,Ty)}} = \left|\frac{1}{p+1} - \frac{1}{q+1}\right|^{\left|\dfrac{1}{\frac{1}{p+1} - \frac{1}{q+1}}\right|}$$

$$= \left[\frac{q-p}{(p+1)(q+1)}\right]^{\dfrac{(p+1)(q+1)}{q-p}},$$

while on the other hand

$$d(x, y)^{\dfrac{1}{d(x,y)}} = \left|\frac{1}{p} - \frac{1}{q}\right|^{\dfrac{1}{\left|\frac{1}{p} - \frac{1}{q}\right|}}$$

$$= \left[\frac{q-p}{(qp)}\right]^{\dfrac{(qp)}{q-p}},$$

we now show that

$$\left[\frac{q-p}{(p+1)(q+1)}\right]^{\dfrac{(p+1)(q+1)}{q-p}} \leq \frac{1}{2}\left[\frac{q-p}{(qp)}\right]^{\dfrac{(qp)}{q-p}}$$

or equivalently

$$\left[\frac{q-p}{(p+1)(q+1)}\right]^{\dfrac{(p+q+1)}{q-p}}\left[\frac{qp}{(p+1)(q+1)}\right]^{\dfrac{(qp)}{q-p}} \leq \frac{1}{2}.$$

This last inequality is indeed true because

$$\left[\frac{pq}{(p+1)(q+1)}\right]^{\dfrac{(pq)}{q-p}} \leq 1$$

We also have $pq < q+1$, $\dfrac{pq}{q-p} < 0$ and

$$. \tag{2.13}$$

Since, for all $p, q \in \mathbb{N}$ we have $q \leq 3p + pq + 1$, therefore $2(q-p) \leq (p+1)(q+1)$ and hence the base in (2.13) is lesser than $\dfrac{1}{2}$ and the exponent is is greater than one (for all $p, q \in \mathbb{N}$, $p+q+1 > q-p$ is trivially satisfied.) On the other hand, taking $x = \dfrac{1}{p}$ and $y = 0$ we have

$$d(Tx,Ty)^{\dfrac{1}{d(Tx,Ty)}} = \left[\frac{1}{p+1}\right]^{p+1} \leq \frac{1}{2}\left[\frac{1}{p}\right]^{p} = \frac{1}{2}d(x,y)^{\dfrac{1}{d(x,y)}},$$

because

$$\left(\frac{p}{p+1}\right)^{p}.\frac{1}{p+1} \leq \frac{1}{2}$$

Since $\dfrac{p}{p+1} \leq \dfrac{1}{2}$ and $\dfrac{1}{p+1} \leq \dfrac{1}{2}$, therefore T satisfies (2.11).

2.3 Fixed Point Theorems in Partially Ordered Metric Spaces

Ran and Reuring [79] got the insipration from Turinici [95] and proved some fixed point results for mappings on metric spaces (X, d) which are contractive for those pairs of points from X which are related. Fascinatingly, many results have been established by various authors on ordered sets equipped with complete metric space [1, 32, 39, 68–70, 76, 78, 79, 98]. These intresting results are the fusion of Banach principle and Knaster-Tarski's theorem [40, 47]. Here we include some basic definitions.

Definition 2.3.1.
A set X is said to be a "partially ordered set" with some binary relation $\preceq$ if for all $a, b, c \in X$, the following conditions are satisfied:

(i) $a \preceq a$, *(reflexivity)*

(ii) $a \preceq b$ and $b \preceq c \Rightarrow a \preceq c$, *(transitivity)*

(iii) $a \preceq b$ and $b \preceq a \Rightarrow a = b$. *(anti-symmetric)*

Definition 2.3.2.
Let $T : X \to X$ be a self map and $(X, \preceq)$ be a partially ordered set. The mapping T is ***non increasing*** if

$$a_1 \preceq a_2 \Rightarrow T(a_1) \succeq T(a_2) \qquad \text{for all } a_1, a_2 \in X.$$

The mapping T is ***non decreasing*** if

$$a_1 \preceq a_2 \Rightarrow T(a_1) \preceq T(a_2) \qquad \text{for all } a_1, a_2 \in X.$$

If

$$a_1 \preceq a_2 \Rightarrow T(a_1) \succeq T(a_2) \text{ or } T(a_1) \preceq T(a_2) \text{ for all } a_1, a_2 \in X,$$

then we say that the related elements have the related images under the mapping T.

Definition 2.3.3.
An operator $T : X \to X$ is termed as a "Picard operator" if

$$\lim_{n\to\infty} T^n a = a_* \text{ for all } a \in X,$$

where X is a space induced with a metric d provided that a_* is unique fixed point of T.

Example 2.3.4.
Let $\Omega = [0, \infty]$ and $T : \Omega \to \Omega$ be defined by

$$T\omega = \frac{\omega}{2}.$$

Then

$$T^n\omega = \frac{\omega}{2^n},$$

which implies that T is a Picard operator since

$$\lim_{n\to\infty} T^n\omega = 0 \text{ for all } \omega \in \Omega.$$

Recently a number of results have been reported with some adequate conditions for $T : X \to X$ to be a Picard operator where (X, d) is a metric space endowed with a partial ordering $\preceq$. Ran and Reurings [79] took the initiative in this direction by giving the following result. They furnished their work by providing its application to linear matrix equations, as well as the nonlinear ones.

Theorem 2.3.5. [79]
"Consider a complete metric space (X, d), where X is a partially ordered set endowed with a partial ordering $\preceq$. Let $T : X \to X$ satisfies:

$$d(Ta_1, Ta_2) \preceq kd(a_1, a_2) \text{ for all } a_1,\ a_2 \in X, \text{ with } a_1 \preceq a_2,\ \ k \in (0, 1).$$

Furthermore, if the mapping T satisfies the following conditions :

(i) there exists $a_0 \in X$ with $a_0 \preceq Ta_0$ or $Ta_0 \preceq a_0$,

(ii) T is monotone and continuous,

(iii) each pair from the elements of X has an upper and a lower bound,

then T is a Picard operator."

Theorem 2.3.5 is generalized by Nieto and Rodriguez-Lopez [68] in the following way.

Theorem 2.3.6. [68]
"Consider a partially ordered set $(X, \preceq)$. Let d be a complete metric define on X. Let T be a non decreasing self map with the property

$$d(Ta_1, Ta_2) \preceq \kappa d(a_1, a_2) \text{ for all } a_1, a_2 \in X, \text{ with } a_1 \preceq a_2, \text{ and } \kappa \in (0, 1).$$

Also, suppose that T satisfies one of the following assertions:

(i) for some $a_0 \in X$ such that either $a_0 \preceq Ta_0$ or $Ta_0 \preceq a_0$ and T is continuous,

(ii) for any non decreasing sequence $\{a_n\}, a_n \to a$ implies $a_n \preceq a$ for $n \in \mathbb{N}$ and there exists $a_0 \in X$ such that $a_0 \preceq Ta_0$,

(iii) for any non increasing sequence $\{a_n\}, a_n \to a$ implies $a \preceq a_n$ for $n \in \mathbb{N}$ and there exists $a_0 \in X$ such that $Ta_0 \preceq a_0$,

then T has a fixed point. Furthermore, T is a Picard operator if each pair of elements of X has an upper or lower bound."

Working on the same guidelines the authors in ([68],[71], [76], [88]) extended above result by enervating continuity condition in a more general way.

Theorem 2.3.7.
"Let X be a partially ordered set having a partial ordering $\preceq$, with (X, d) being a complete metric space. Suppose that $T : X \to X$ be a map which preserves related elements and also satisfies

$$d(Ta_1, Ta_2) \preceq \kappa d(a_1, a_2) \text{ for all } a_1, a_2 \in X, \text{ with } a_1 \preceq a_2, \text{ and } \kappa \in (0, 1).$$

Moreover, T satisfies the following assertions:

(i) T is either orbitally continuous, or
for any sequence $a_n \to a$, each pair of elements (a_n, a_{n+1}) is related. Then there exists a subsequence $\{a_{n_k}\}$ for $n \in \mathbb{N}$ such that the pair of elements (a_{n_k}, a) are related for $k \in \mathbb{N}$,

(ii) there exists an element a_0 in X such that the pair (a_0, Ta_0) is related,

then T has a fixed point.
Moreover, T is a Picard operator if each pair from the elements of X has an upper or lower bound ."

2.4 Fixed Point Theorems in Metric Spaces Endowed with a Graph

Jachymski [46] generalized the idea of partial ordering by utilizing the graph theory. He unified and extended the results given by the authors [69, 76, 79]. Jachymski considered a map $T : X \to X$ where (X, d) is a complete metric space and proved that it has a fixed point even if the mapping satisfies the contraction condition for those pairs of points which form the edges of a graph. The following definitions and theorems by Jachymski [46] are quite supportive for us in achieving some important results in the next chapters.

Definition 2.4.1. [46]
"Let (X, d) be a metric space and G be a graph. A mapping $T : X \to X$ is called ***Banach G-contraction*** or simply G-contraction if T is edge preserving, i.e.,

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G) \text{ for all } x, y \in X, \tag{2.14}$$

and also there exist $k \in (0, 1)$ such that

$$d(Tx, Ty) \preceq kd(x, y) \text{ for all } x, y \in X \text{ with } (x, y) \in E(G)." \tag{2.15}$$

Example 2.4.2.
Let $T : \Omega \to \Omega$ be a constant function. It can be easily verified that T is a Banach G -contraction since $E(G)$ contains all loops (in fact $E(G)$ must contain all loops if we wish any constant function to be a G-contraction.)

Example 2.4.3.
Let $T : \Omega \to \Omega$ be a Banach contraction. Then T is a G-contraction, when we consider the graph $G_\circ$ with

$$E(G_0) := \Omega \times \Omega.$$

Example 2.4.4.
Let $\Omega = (0, 1)$ and d be the usual metric defined as

$$d(\omega_1, \omega_2) = |\omega_1 - \omega_2|.$$

Consider a graph G with

$$V(G) = \Omega$$

and

$$E(G) = \{(\omega, \omega) : \omega \in \Omega\} \cup \{\left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right) : n \in \mathbb{N})\},$$

then $T : \Omega \to \Omega$ defined as

$$T\omega = \frac{\omega}{2}$$

is a Banach G-contraction.

Example 2.4.5.
Let (Ω, d) be a metric space with a partial order $\preceq$. consider a graph G_1 for which

$$E(G_1) = \{(\omega_1, \omega_2) \in \Omega \times \Omega : \omega_1 \preceq \omega_2\}. \tag{2.16}$$

For this graph, (4.10) means T is non decreasing with respect to this order. Recently, this class of G_1-contraction was studied by Neito and Rodriguez-Lopez [68].

Example 2.4.6.
Consider a metric space (Ω, d) equipped with a partial order $\preceq$. Let us construct the graph G_2 by

$$E(G_2) = \{(\omega_1, \omega_2) \in \Omega \times \Omega : \omega_1 \preceq \omega_2 \ \vee \ \omega_2 \preceq \omega_1\}. \tag{2.17}$$

In general, graph $G = G_2$ means that T maps comparable elements onto comparable elements, so class of G_2-contractions coincides with the class considered by Petrusel and Rus and Neito [76] and Rodriguez-Lopez [68].

Definition 2.4.7.
Jachymski [46] declares that "a mapping $T : X \to X$ is called ***orbitally continuous*** if for all $x, y \in X$ and any sequence $\{k_n\}_{n \in \mathbb{N}}$ of positive integers,

$$T^{k_n}x \to y \quad \text{implies} \quad T(T^{k_n})x \to Ty \quad \text{as} \quad n \to \infty."$$

Definition 2.4.8. Jachymski [46]
"A self map T on X is called G-***continuous*** if each $x \in X$ there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$, such that

$$\{x_n\} \text{ converges to } x \text{ and } (x_n, x_{n+1}) \text{ belongs to } E(G) \text{ for } n \in \mathbb{N} \Longrightarrow Tx_n \to Tx."$$

Definition 2.4.9. Jachymski [46]
"A self map $T : X \to X$ is said to be ***orbitally*** G-***continuous*** if for each $x, y \in X$

and any sequence $\{k_n\}_{n\in\mathbb{N}}$,

$$T^{k_n}x \to y \text{ and } (T^{k_n}x, T^{k_{n+1}}x) \in E(G) \text{ for } n \in \mathbb{N} \implies T(T^{k_n})x \to Ty.\text{''}$$

Example 2.4.10.
Let $X = (0,1)$ and d be the usual metric defined as

$$d(x,y) = |x - y|.$$

Consider a graph G with $V(G) = X$ and

$$E(G) = \{(\frac{1}{2^n}, \frac{1}{2^{n+1}}) : n \in \mathbb{N}\}.$$

Define a map $T : X \to X$ by

$$Tx = \frac{x}{2}$$

then $T\frac{1}{2^n} \to T(0)$ where $\left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right) \in E(G)$. Hence T is orbitally G-continuous.

Subsequently in this section (X,d) is a metric space, and G is a directed graph with X as a set of vertices and $E(G) \supseteq \Delta$. We imitate the following result from ([46] Theorem 3.2) proved by Jachymski.

Theorem 2.4.11.
" Let (X,d) be a complete metric space escorted with a graph G. Assume that the tuple (X,d,G) satisfies the following property:
$(\mathcal{P})$: for any $\{x_n\}_{n\in\mathbb{N}}$, if $x_n \to x$ with $(x_n, x_{n+1}) \in E(G)$ there is a sub-sequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.
Consider a G-contraction $T : X \to X$, and

$$X_T := \{x \in X : (x, Tx) \in E(G)\}.$$

Then T satisfies the following assertions:

1. card$\mathcal{F}$ix(T) =card $\{[x]_{\tilde{G}} : x \in X_T\}$.
2. $\mathcal{F}\text{ix}(T) \neq \phi \Leftrightarrow X_T \neq \phi$.
3. T has a unique fixed point iff there exists $x_\circ \in X_T \subseteq [x_\circ]_{\tilde{G}}$.
4. $T|_{[x]_{\tilde{G}}}$ is a Picard operator for any $x \in X_T$.

5. T is a Picard operator if $X_T \neq \phi$ and G is weakly connected.

6. If $X^{'} := \cup\{[x]_{\tilde{G}} : x \in X_T\}$, then $T|_{X'}$ is a weakly Picard operator.

7. T is a weakly Picard operator if $T \subseteq E(G)$."

Samreen and Kamran [86] also obtained some motivating fixed point results for Banach type contractions escorted with the graph G, by using the following condition, which is definitely weaker than $(\mathcal{P})$.

$(\mathcal{P}')$: "for any sequence $\{T^n y\}$ in X such that $T^n y$ converges to $y_0 \in X$ with $(T^{n+1}y, T^n y) \in E(G)$ there exist a subsequence $\{T^{n_k}y\}$ of $\{T^n y\}$ and $n_0 \in \mathbb{N}$ for which $(y_0, T^{n_k}y) \in E(G)$ $\forall$ $n_k \in \mathbb{N}$ for all $k \geq n_0$."

Example 2.4.12. [86]
Let (X, d) be a metric space with $X = [0, 1]$ and d a usual metric. Let us consider a graph, G set of whose vertices is the set X and

$$E(G) = \left\{\left(\frac{n}{n+1}, \frac{n+1}{n+2}\right) : n \in \mathbb{N}\right\}$$

$$\cup \left\{\left(\frac{x}{2^n}, \frac{x}{2^{n+1}}\right) : n \in \mathbb{N}, x \in [0,1]\right\}$$

$$\cup \left\{\left(\frac{x}{2^n}, 0\right) : n \in \mathbb{N}, x \in [0,1]\right\}.$$

Consider a sequence $y_n = \{\frac{n}{n+1}\}$ in X then $(y_n, y_{n+1}) \in E(G)$. But

$$\frac{n}{n+1} \to 1 \text{ and } (1, \frac{n}{n+1}) \notin E(G)$$

implies that G does not satisfy $(\mathcal{P})$, while when we define $T : X \to X$ by $Tx = \dfrac{x}{2}$, it fulfills $(\mathcal{P}')$ since

$$T^n x = \frac{x}{2^n} \to 0 \text{ as } n \to \infty.$$

Definition 2.4.13.
For a complete metric space (X, d) escorted with a graph G, a self map T on X is called "weakly G-contractive" if :

1. $(Tx_1, Tx_2) \in E(G)$ whenever $(x_1, x_2) \in E(G)$ for all $x_1, x_2 \in X$,

2. $d(Tx_1, Tx_2) \leq d(x_1, x_2) - \psi(d(x_1, x_2))$ whenever $(x_1, x_2) \in E(G)$,

where ψ is a self mapping on $[0, \infty]$ which is non decreasing and continuous. Also ψ has positive images on $(0, \infty)$ with $\psi(0) = 0$.

Example 2.4.14.
Let G_0 be a graph defined by

$$G = (X, X \times X).$$

Then it can be easily seen that any weakly contractive map is G-weakly contractive.

Theorem 2.4.15. [46]
"Consider a complete metric space (X, d) accompanied with a graph G. Also assume that T is a weakly G-contractive self mapping on X which satisfies the following assertions:

1. G satisfies the property $(\mathcal{P}')$,
2. there exists some $a_0 \in X_T := \{a \in X : (a, Ta) \in E(G)\}$.

Then there exists a unique fixed point $a^* \in [a_0]_{\tilde{G}}$ for $T|_{[a]_{\tilde{G}}}$.
Moreover, $T^n b \to a^*$ for any $b \in [a_0]_{\tilde{G}}$."

2.5 Fixed Point Theorems in $b-$metric Spaces

Fixed point theory has been established in various directions for example by improving the contaction conditions and similarly, by changing the space, by various abstract spaces. Now we will discuss b-metric space which is a motivating generalization of a metric space. Bourbaki [19], Bakhtin [10], Czerwik [29, 30] and Heinonen [43] instigated b-metric spaces in their work. Later on, many authors have contributed a lot in this direction for single valued and multivalued mappings. [8, 10, 13, 15–19, 30, 43, 72, 75]

Definition 2.5.1. [30]
"A function $d : X \times X \to \mathbb{R}^+$ is called a b-metric on X with $X \neq \emptyset$ if it satisfies the following conditions:

(B1) $d(x_1, x_2) = 0 \quad \Leftrightarrow \quad x_1 = x_2$,

(B2) $d(x_1, x_2) = d(x_2, x_1) \ \ \forall \ \ x_1, x_2 \in X$,

(B3) $d(x_1, x_2) \leq s\,(d(x_1, x_3) + d(x_3, x_2)) \ \ \forall \ \ x_1, x_2, x_3 \in X$,

where $s \geq 1$ is a real number. The pair (X, d) is called a b-metric space."

It seems important to perceive that when $s = 1$, the triangular inequality of metric space is satisfied. But it is obviously not true if $s > 1$. Thus the set of b-metric spaces is commendably bigger than that of ordinary metric spaces. We give the following examples to justify our remark.

Example 2.5.2. [64]
Let $X = \{-1, 0, 1\}$. Define $d_s : X \times X \to \mathbb{R}^+$ by

$$d_s(x_1, x_2) = d_s(x_2, x_1) \, \forall \, x_1, x_2 \in X, \;\; d_s(x_1, x_1) = 0 \ \ \forall \, x_1 \in X.$$

Also

$$d_s(-1, 1) = 3, d_s(-1, 0) = d_s(0, 1) = 1.$$

In fact, we have

$$d_s(-1, 0) + d_s(0, 1) = 1 + 1 = 2 < 3 = d_s(-1, 1) = 3,$$

which implies that the triangular inequality is not satisfied which implies that d_s is not a metric space. Since

$$d_s(-1, 1) = \frac{3}{2}\,(d_s(-1, 1) + d_s(0, 1)) \text{ here } \ s = \frac{3}{2},$$

hence d_s is a b-metric space.

Example 2.5.3. [83]
Let (X, d) be a metric space and

$$\rho(x, y) = (d(y, x))^p \ \ \forall \, x, y \in X,$$

where $p > 1$ is real number. Then ρ is a b-metric space with $s = 2^{p-1}$.

Remark 2.5.4. [21]
Let (X, d) be a b-metric space, then the following conditions are satisfied:

(i) every convergent sequence is a Cauchy and has a unique limit,

(ii) generally a b-metric space is not continuous.

Theorem 2.5.5. [65]
"Consider a b-metric space (X, d) accompanied with a graph G. Let $T : X \to X$ be a mapping such that

$$d(Tx_1, Tx_2) \leq kd(x_1, x_2) \tag{2.18}$$

for all $x_1, x_2 \in X$ with (x_1, x_2) belonging to $E(\tilde{\mathcal{G}})$, where $k \in (0, \frac{1}{s})$ is a constant. Moreover, assume that the b-metric space (X, d) with the graph G has the following properties:

(P1) if for a sequence $\{x_n\} \subseteq X \ni x_n \to x$ with $(x_n, x_{n+1}) \in E(\tilde{G})\ \forall\ n \geq 1$, we have a subsequence x_{n_k} of x_n such that

$$(x_{n_k}, x) \in E(\tilde{\mathcal{G}})\ \forall\, k \geq 1,$$

(P2) if x_1, x_2 are the fixed point of T in X then $(x_1, x_2) \in E(\tilde{\mathcal{G}})$,

(P3) the set C_T which is the collection of all $x \in X$ such that $(T^n x, T^m x) \in E(\tilde{\mathcal{G}})$ for $m, n = 0, 1, 2, \cdots$ is non empty,

then T has a unique fixed point."

Corollary 2.5.6.
"Let (X, d) be a b-metric space and a mapping $T : X \to X$ be such that (2.18) holds for all $x, y \in X$, where $k \in (0, \frac{1}{s})$ is a constant. Then T has a unique fixed point x_0 in X and $T^n x \to x_0$ for all $x \in X$."

This Corollary follows from Theorem 2.5.5 by putting $G = G_0$.

Chapter 3

Fixed Point Theorems for C^*-valued Contractions

Fixed point theory is the center of focus for many mathematicians from last few decades. A lot of generalizations of Banach fixed point theorem had been introduced. Recently, Ma et al. [63] gave an interesting generalization of this principle. In that article, the authors replaced the underlined metric space by the algebra valued metric space. This is a notable contribution towards fixed point theory.

3.1 C^*-valued Contractions

In this section, we will first refresh the idea of a C^*-valued metric space. We will also provide some illustrative examples to show that C^*-valued metric space generalizes the concept of metric space by replacing the set of reals by an algebra $\mathbb{A}$.

Definition 3.1.1. [63]
"Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{A}$ is called a C^*-valued metric on X if it satisfies the following conditions:

i) $d(x, y) \succeq \theta$ for all $x, y \in X$,

ii) $d(x, y) = \theta \Leftrightarrow x = y$,

iii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

iv) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$,

where $\preceq$ is defined in (1.2) and θ be the zero element of the algebra $\mathbb{A}$. The tuple $(X, \mathbb{A}, d)$ is called a C^*-valued metric space."

Note that if we choose $\mathbb{A} = \mathbb{R}$ then d is a metric space.

Definition 3.1.2. [63]
"Let $(X, \mathbb{A}, d)$ be a C^*-valued metric space and $x \in X$. A sequence $\{x_n\}$ in $(X, \mathbb{A}, d)$ is said to be ***convergent*** with respect to $\mathbb{A}$, if for any $\epsilon > 0$, there exists a positive integer N such that

$$\|d(x_n, x)\| \leqslant \epsilon \text{ for all } n > N."$$

Definition 3.1.3. [63]
"A sequence $\{x_n\}$ is called a ***Cauchy sequence*** with respect to $\mathbb{A}$ if for any $\epsilon > 0$, there exists a positive integer N such that

$$\|d(x_n, x_m)\| \leqslant \epsilon \text{ for all } n, m > N.$$

If every Cauchy sequence with respect to $\mathbb{A}$ is convergent, then $(X, \mathbb{A}, d)$ is said to be a ***complete*** C^****-valued metric space***."

Example 3.1.4.
If we take $X = \mathbb{R}$ and $\mathbb{A} = \mathbb{R}^2$, and define $d : X \times X \to \mathbb{A}$ as:

$$d(x,y) = (0, |x-y|),$$

then $(X, \mathbb{A}, d)$ is a complete C^*-valued metric space.

Example 3.1.5.
Let $X = \mathbb{R}$ and $\mathbb{A} = M_{2\times 2}(\mathbb{R})$. Define $d : X \times X \to \mathbb{A}$ as:

$$d(x,y) = \begin{pmatrix} |x-y| & 0 \\ 0 & \alpha|x-y| \end{pmatrix}, \tag{3.1}$$

where $x, y \in \mathbb{R}$ and $\alpha \geq 0$ is a constant. It is easy to verify that d is a complete C^*-valued metric space by completeness of $\mathbb{R}$.

3.2 Fixed Point Theorems for C^*-valued Contractions

Development in metric fixed point theory is based on two things. On one side, the usual contractive conditions are replaced by weak contractions and on the other side, action spaces are modified. Recently, O'Regan and Petrusel [80], and Caballero et al. [25] worked on ordered metric spaces. In 2007 Huang and Zhang [45] introduced the concept of cone metric space by replacing real numbers by a Banach space equipped with a partial oreder and proved some fixed point theorems. Ma et al. [63] further improved this idea by giving the concept of C^*- valued contraction. In this section, we will first debate on the C^*- valued contraction by providing different examples. Later, we will imitate the fixed point results given by Ma et al. These results are stunning generalizations of the fixed point results on metric space over the set of reals.

Definition 3.2.1. [63]
"Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space. A mapping $T : X \to X$ is said to be a C^****-valued contractive*** mapping on X if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx, Ty) \preceq A^* d(x, y)A, \quad \text{for all } \; x, y \in X.\text{"} \tag{3.2}$$

Example 3.2.2.
Let $X = \mathbb{R}$ and $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ with

$$\|A\| = \sqrt{\sum_{i,j=1}^{2} |a_{ij}|^2} \;\text{ and }\; A^* = A,$$

where a_{ij} are the entries of A. Define $d : X \times X \to \mathbb{A}$ as:

$$d(x, y) = \begin{pmatrix} |x-y| & 0 \\ 0 & |x-y| \end{pmatrix}, \tag{3.3}$$

where $x, y \in \mathbb{R}$. Define a mapping $T : X \to X$ by

$$Tx = \frac{x}{4},$$

then T is a C^*- valued contraction, since

$$d(Tx,Ty) = \begin{pmatrix} |\frac{x-y}{4}| & 0 \\ 0 & |\frac{x-y}{4}| \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} |x-y| & 0 \\ 0 & |x-y| \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$= A^* d(x,y)A,$$

with $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and $\|A\| < 1$.

We, now, state a recently reported result on the existence of a unique fixed point by Ma et al. [63]. We have generalized this result in Chapter 4.

Theorem 3.2.3. [63]
"If $(X, \mathbb{A}, d)$ is a C^*-algebra valued metric space and T satisfies (3.2), then T has a unique fixed point in X."

Definition 3.2.4. [63]
"Let X be a non empty set. A self mapping T on X is called a "C^*-algebra valued expansion mapping" on X, if $T : X \to X$ satisfies the following:

(1) T is onto mapping,

(2) $d(Tx_1, Tx_2) \succeq A^* d(x_1, x_2)A$, forall $x_1, x_2 \in X$,

where $A \in \mathbb{A}$ is an non singular element with $\|A^{-1}\| \leq 1$."

Example 3.2.5. Let $X = \mathbb{R}$ and $\mathbb{A} = \mathbb{R}^2$. Define $d : X \times X \to \mathbb{A}$ as:

$$d(x_1, x_2) = (|x_1 - x_2|, |x_1 - x_2|)\,.$$

Also define

$$\|A\| = \text{Sup}\{|a_1|, |a_2|\}, \text{ and } A^* = A \ \ \forall A \in \mathbb{A}.$$

Consider a self mapping T on X which is defined by

$$Tx = 2x.$$

Then,

$$d(Tx_1, Tx_2) = d(2x_1, 2x_2)$$

$$= \left(\sqrt{2}, \sqrt{2}\right)(|x_1 - x_2|, |x_1 - x_2|)\left(\sqrt{2}, \sqrt{2}\right)$$

with $A^{-1} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\|A^{-1}\| < 1$. Hence T is a "C^*-algebra valued expansion mapping" on X.

Following are two important fixed point theorems given by Ma et al. [63]

(T1) "Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space. Then for the expansion mapping T, there exists a unique fixed point."

(T2) "Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space. Suppose the mapping $T : X \to X$ satisfies

$$d(Tx, Ty) \preceq A\left(d(Tx, y) + d(Ty, x)\right), \tag{3.4}$$

where $A \in \mathbb{A}'$ and

$$\mathbb{A}' = \{a \in \mathbb{A} : ab = ba, \ \forall\, b \in \mathbb{A}\}$$

and $\|A\| \leq \dfrac{1}{2}$. Then there exists a unique fixed point in X."

Chapter 4

Fixed Point Theorems for C^*-valued G-Contractions

Jachymski [46] generalized some fixed point results by using an innovative idea of establishing these results on the setting of a graph. Motivated by his work, we extend and improve the result of Ma et al. [63] on the setting of graph. Our result generalizes and extends the main result of Jachymski and Ma et al. [63] and those contained therein. We also present some examples to elaborate our new notions.

4.1 C^*-valued G-contraction

In this section, we first elaborate the idea C^*-valued G-contractions and then provide some examples to show the generality of our new definition.

Definition 4.1.1.
Suppose $(X, \mathbb{A}, d)$ be a C^*-valued metric space endowed with the graph $G = (V(G), E(G))$. A mapping $T : X \to X$ is called a C^****-valued*** G***-contraction*** on X, if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx, Ty) \preceq A^* \, d(x, y) \, A, \qquad \forall \, (x, y) \in E(G). \tag{4.1}$$

Remark 4.1.2.
By taking $G_1 = (X, X \times X)$, we see that a C^*-valued contraction is a C^*-valued G_1-contraction.

Following example shows that converse of the above statement is not true in general.

Example 4.1.3.
Consider the algebra, $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ of all 2×2 matrices with usual operations of addition, scalar multiplication and matrix multiplication. Note that

$$\|A\| = \sqrt{\sum_{i,j=1}^{2} |a_{ij}|^2}$$

defines a norm on $\mathbb{A}$ and $* : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$, given by $A^* = A$, defines a convolution on $M_{2\times 2}(\mathbb{R})$. Thus $\mathbb{A}$ becomes a C^*-algebra. For

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_{2\times 2}(\mathbb{R}),$$

we say

$$A \preceq B \text{ if and only if } A - B \succeq \theta. \tag{4.2}$$

It is straightforward to see that $\preceq$ given by (4.2) is a partial order on $M_{2\times 2}(\mathbb{R})$. Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{A}$ by

$$d(x, y) = \begin{pmatrix} |x-y| & 0 \\ 0 & |x-y| \end{pmatrix}. \tag{4.3}$$

It is easy to check that d satisfies all conditions of Definition 3.1.1. Therefore, $(\mathbb{R}, \mathbb{A}, d)$ is a C^*-valued metric space. Define $T : \mathbb{R} \to \mathbb{R}$ by

$$Tx = \frac{x^2}{3},$$

and consider the graph $G = (V(G), E(G))$, where $V(G) = \mathbb{R}$ and

$$E(G) = \left\{ \left(\frac{1}{3^n}, \frac{1}{3^{2n+1}}\right) : n = 1, 2, \cdots \right\} \cup \Big\{(x, x) : x \in \mathbb{R}\Big\}. \tag{4.4}$$

Note that, for each $n \in \mathbb{N}$,

$$\left(T\frac{1}{3^n}, T\frac{1}{3^{2n+1}}\right) = \left(\frac{1}{3^{2n+1}}, \frac{1}{3^{4n+3}}\right) \in E(G).$$

Also, for each $x \in \mathbb{R}$, $(Tx, Tx) = \left(\frac{x^2}{3}, \frac{x^2}{3}\right)$, which is again an edge in the graph G. Moreover,

$$d\left(T\frac{1}{3^n}, T\frac{1}{3^{2n+1}}\right) = \begin{pmatrix} \left|\frac{1}{3^{2n+1}} - \frac{1}{3^{4n+3}}\right| & 0 \\ 0 & \left|\frac{1}{3^{2n+1}} - \frac{1}{3^{4n+3}}\right| \end{pmatrix}$$

$$\preceq \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \left|\frac{1}{3^n} - \frac{1}{3^{2n+1}}\right| & 0 \\ 0 & \left|\frac{1}{3^n} - \frac{1}{3^{2n+1}}\right| \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$
$$\preceq A^* d\left(\frac{1}{3^n}, \frac{1}{3^{n+1}}\right) A$$

where

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \text{ with } \|A\| < 1.$$

Hence the contractive condition (4.1) holds for all edges that belong to the graph G.

Furthermore, the contractive condition (3.2) is not satisfied, for example, at $x = 1,\ y = 7$. Hence T is C^*-valued G-contraction but not a C^*-valued contraction.

4.2 Fixed Point Theorems for C^*-valued Banach G-Contractions

It is a natural question to ask whether the mapping T, considered above, has a fixed point if the contraction condition holds for those pair of elements that form edges of the graph as defined by Jachmjski [46]. In this section, we give positive answer to this question by proving a fixed point theorem for such contractions. We construct some examples to elaborate the generality of our notion and result.

Lemma 4.2.1.

Let $\mathbb{A}$ be a C^*-algebra and $x \in \mathbb{A}$ such that $\|x\| < 1$, then

$$\lim_{m\to\infty} \sum_{k=m}^{n} \|x\|^k = 0. \tag{4.5}$$

Proof.

Proof of (4.5) follows from the fact that $\sum_{k=m}^{n} \|x\|^k$ is a geometric series with common ratio $\|x\| < 1$ and $m \to \infty$ implies $\|x\|^m \to 0$. □

Theorem 4.2.2.

Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space endowed with the graph G . Suppose $T : X \to X$ is a C^*-valued-G-contraction on X satisfying the property:

($\mathcal{P}'$): for any $\{T^n x\}$ in X such that $T^n x \to y \in X$ with $(T^{n+1}x, T^n x) \in E(G)$, there exists a subsequence $\{T^{n_k}x\}$ of $\{T^n x\}$ and $n_0 \in \mathbb{N}$ such that $(y, T^{n_k}x) \in E(G)$ for all $k \geq n_0$;

and the following conditions:

I) if $(x, y) \in E(G)$ then $(Tx, Ty) \in E(G)$;

II) there exists an $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$.

Then T has a fixed point.
Moreover, if y, z are two fixed points of T and $(y, z) \in E(G)$, then $y = z$.

Proof.

It is clear that if $A = \theta$ then T maps X into a single point, since

$$\theta \preceq d(Tx, Ty) \preceq \theta \text{ for all } (x, y) \in E(G).$$

Thus without loss of generality, we assume that

$$A \neq \theta.$$

From hypothesis (II), we have an $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, then by using assumption (I), we get $(Tx_0, T^2x_0) \in E(G)$. Continuing in the same way, we get a sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n = T^{n+1}x_0$$

and

$$(x_{n-1}, x_n) \in E(G) \text{ for all } n \in \mathbb{N}.$$

Let us denote $d(x_0, x_1)$ by $P \in \mathbb{A}$. From (4.1), we have

$$\begin{aligned} d(x_{n+1}, x_{n)} &\preceq A^* d(x_n, x_{n-1}) A \\ &\preceq (A^*)^2 d(x_{n-1}, x_{n-2}) A^2 \\ &\vdots \\ &\preceq (A^*)^n d(x_0, x_1) A^n \\ &= (A^*)^n P A^n. \end{aligned}$$

For $n+1 > m$, we get

$$\begin{aligned} d(T^{n+1}x_0, T^m x_0) &= d(x_{n+1}, x_m) \\ &\preceq d(x_{n+1}, x_n) + d(x_n, x_{n-1}) + ... + d(x_{m+1}, x_m) \\ &= \sum_{k=m}^{n} (A^*)^k P A^k \\ &= \sum_{k=m}^{n} (P^{\frac{1}{2}} A^k)^* (P^{\frac{1}{2}} A^k) \\ &= \sum_{k=m}^{n} |P^{\frac{1}{2}} A^k|^2 \\ &\preceq \sum_{k=m}^{n} \||P^{\frac{1}{2}} A^k|^2\| I \\ &= \|P^{\frac{1}{2}}\|^2 \sum_{k=m}^{n} \|A^{2k}\| I. \end{aligned}$$

Since $\|A\| < 1$, it follows from Lemma 4.2.1 that

$$d(T^{n+1}x_0, T^m x_0) \to \theta \text{ as } \ m \to \infty.$$

This shows that $(T^n x_0)$ is a Cauchy sequence with respect to $\mathbb{A}$. Further, completeness of $(X, \mathbb{A}, d)$ implies that there exists $y \in X = V(G)$ such that

$$\lim_{n\to\infty} T^n x_0 = y.$$

As $T^n x_0 \to y$ and $(T^{n+1}x, T^n x) \in E(G)$ for all $n \in \mathbb{N}$, therefore by property $(\mathcal{P}')$, there exists a subsequence $(T^{n_k} x_0)$ and $n_0 \in \mathbb{N}$ such that $(T^{n_k} x_0, y) \in E(G)$ for all $k \geq n_0$. It follows that

$$\begin{aligned}\theta &\preceq d(Ty, y)\\ &\preceq d(Ty, T^{n_{k+1}} x_0) + d(T^{n_{k+1}} x_0, y)\\ &\preceq A^* d(y, x_{n_k+1}) A + d(x_{n_{k+2}}, y) \to \theta \quad \text{as} \quad k \to \infty.\end{aligned}$$

Thus

$$Ty = y.$$

Suppose $z \in V(G)$ be another fixed point of T such that $(z, y) \in E(G)$ then,

$$\begin{aligned}0 &\leq \|d(z, y)\|\\ &= \|d(Tz, Ty)\|\\ &\leq \|A^* d(z, y) A\|\\ &\leq \|A\|^2 \, \|d(z, y)\|\\ &< \|d(z, y)\|,\end{aligned}$$

since $\|A\| \leq 1$. This is possible only if $\|d(z, y)\| = 0$. This implies $d(z, y) = \theta$. Hence $z = y$. □

Remark 4.2.3.
By taking $G = (X, X \times X)$, we see that Theorem 3.2.3 is a special case of Theorem 4.2.2. Moreover, [46, Theorem 3.2] becomes special case of Theorem 4.2.2 when $\mathbb{A} = \mathbb{R}$.

Example 4.2.4.
Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{R}$ endowed with the usual metric

$$d(x, y) = |x - y|.$$

Consider a graph G consisting of $V(G) = X$ and

$$E(G) = \left\{ (\frac{x}{2^{n-1}}, \frac{x}{2^n}) : n \in \mathbb{N}, x \in X \right\}$$

$$\cup \{(x, x) : x \in X\}$$

$$\cup \left\{ (\frac{x}{2^n}, 0) : n \in \mathbb{N}, x \in X \right\}.$$

Define a mapping $T : X \to X$ by

$$Tx = \frac{x}{2}$$

then

$$x_n = T^n x = \frac{x}{2^n}.$$

By hypothesis, there exists an $x_0 \in X$ such that $(x_0, Tx_0) = (x_0, \frac{x_0}{2}) \in E(G)$. For each $(x, y) \in E(G)$, we have

$$d(Tx, Ty) = d\left(\frac{x}{2}, \frac{y}{2}\right)$$

$$= \left|\frac{x}{2} - \frac{y}{2}\right|$$

$$= \frac{1}{2}|x - y|$$

$$= \frac{1}{\sqrt{2}} d(x, y) \frac{1}{\sqrt{2}}$$

$$= A^* d(x, y) A,$$

where

$$A = A^* = \frac{1}{\sqrt{2}} \in \mathbb{A}.$$

We observe that $T^n x = \dfrac{x}{2^n} \to 0$ as $n \to \infty$ and also $(T^n x, T^{n+1} x) \in E(G)$ for all $n \in \mathbb{N}$. Since all the conditions of Theorem 4.2.2 hold, therefore T has a fixed point.

Example 4.2.5.
Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{R}^2$ with

$$\|a\| = \text{Sup}\{|a_1|, |a_2|\},$$

where

$$a = (a_1, a_2) \in \mathbb{R}^2.$$

Using the partial ordering on $\mathbb{A}$ by $a - b \succeq 0$ if and only if all the entries of $a - b$ are positive where $a, b \in \mathbb{R}^2$. Define a metric $d : X \times X \to \mathbb{A}$ as

$$d(x, y) = (|x - y|,\ |x - y|),$$

endowed with a graph $G = (V(G), E(G))$ where $V(G) = X$ and

$$E(G) = \left\{ \left(\frac{x}{\beta^{n-1}}, \frac{x}{\beta^n} \right) : n \in\ \mathbb{N}, x\ \in X \right\}\ \cup\ \left\{ \left(\frac{x}{\beta^n}, 0 \right) : n \in\ \mathbb{N}, x\ \in X \right\}.$$

Also define a mapping $T : X \to X$ by

$$Tx = \frac{x}{\beta},$$

where $\beta > 2$. For each $(x, y) \in E(G)$, we have

$$d(Tx, Ty) = d\left(\frac{x}{\beta}, \frac{y}{\beta} \right)$$

$$= \left(\left| \frac{x - y}{\beta} \right|, \left| \frac{x - y}{\beta} \right| \right)$$

$$= \left(\frac{1}{\sqrt{\beta}}, \frac{1}{\sqrt{\beta}}\right) (|x-y|,\ |x-y|) \left(\frac{1}{\sqrt{\beta}}, \frac{1}{\sqrt{\beta}}\right)$$

$$= A^* d(x,y) A$$

Here

$$A = A^* = (\frac{1}{\sqrt{\beta}}, \frac{1}{\sqrt{\beta}}) \text{ with } \|A\| < 1.$$

Further, it is easy to see that all the conditions of Theorem 4.2.2 hold. Thus T has a fixed point.

Example 4.2.6.

Let $X = [0,1]$ and $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ endowed with the metric

$$d(x,y) = \operatorname{diag}(|x-y, |x-y|)$$

and

$$\|A\| = \sqrt{\sum_{i,j=1}^{2} |a_{ij}|^2},$$

where a_i are the entries of A. Let G be a graph with $V(G) = X$ and

$$E(G) = \left\{\left(\frac{x}{3^{a_{n-1}}}, \frac{x}{3^{a_n}}\right) : n \in\ \mathbb{N} \text{ and } a_n = 2a_{n-1} + 1 \text{ with } a_0 = 1,\ x\ \in X\right\}$$
$$\cup \left\{(x,x) : x \in X\right\}.$$

Define a mapping $T : X \to X$ by

$$Tx = \frac{x^2}{3}.$$

Then by hypothesis there exists $x_0 = \frac{1}{3} \in X$ such that $(x_0, Tx_0) = (x_0, \dfrac{x_0^2}{3}) \in E(G)$, also $(Tx, Ty) \in E(G)$ for all $(x,y) \in E(G)$.
For each $(x,y) \in E(G)$, we have

$$d(Tx,Ty) = d\left(\frac{x^2}{3}, \frac{y^2}{3}\right)$$

$$= \operatorname{diag}\left(\left|\frac{x^2-y^2}{3}\right|, \left|\frac{x^2-y^2}{3}\right|\right)$$

$$= \frac{1}{3}\text{diag}(|x^2 - y^2|, |x^2 - y^2|)$$

$$= \frac{1}{3}\text{diag}(|x - y||x + y|, |x - y||x + y|)$$

$$\preceq \frac{1}{3}\text{diag}(|x - y|, |x - y|)$$

$$= \text{diag}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{diag}\left(|x - y|,\ |x - y|\right) \text{diag}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= A^* d(x, y) A \quad \text{since } (x, y) \in E(G).$$

Here $A = A^* = \text{diag}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ with $\|A\| < 1$. One can easily verify that

$$x_n = T^n x_0 = \frac{1}{3^{a_n}} \to 0 \text{ as } n \to \infty.$$

Since all the assertions of Theorem 4.2.2 hold, therefore 0 is the fixed point of T.

4.3 Fixed Point Theorems for C^*-valued Caristi Type G-Contractions

Caristi's fixed point theorem ([57], [58]) is a valuable extension of the Banach contraction principle. The proof of "Caristi's fixed point theorem" on metric spaces uses different directions and techniques ([57, 58]). It is also worthmentioning that due to close connection of "Caristi's theorem" with "Ekeland's variational principle [38]," many authors refer it to as Caristi- Ekeland's variational principle. Several authors extended this result on different type of distance spaces for example in [58], Khamsi gave a characterization to the existence of minimal element in a partially ordered set in terms of fixed point of a multivalued map. In this section, we will prove Caristi's fixed point theorem on C^*-valued metric space. We further generalize this version of C^*-valued metric space in the setting of graph.

First of all, we state the following fixed point results on metric space to be used for the further discussion.

Theorem 4.3.1. [38] (Ekeland Variational Principle)

"Consider a complete metric space (X, d) and let $\phi : X \to \mathbb{R}^+$ be a lower semi continuous map. Then

$$x_1 \leq x_2 \text{ if and only if } d(x_1, x_2) \leq \phi(x_1) - \phi(x_2), \quad \forall \quad x_1, x_2 \in X. \tag{4.6}$$

Then $(X, \leq)$ has a minimal element."

Theorem 4.3.2. [24] (Caristi's Fixed Point Theorem)

"Consider a complete metric space (X, d) and let $T : X \to X$ be a self map. Then T has a fixed point provided that there exists a lower semi continuous map $\phi : X \to \mathbb{R}^+$ such that:

$$d(x, Tx) \leq \phi(x) - \phi(Tx) \quad \text{for all } x \in X." \tag{4.7}$$

Now we give some definitions to generalize the above result on C^*-algebra valued metric space.

Definition 4.3.3.
Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space. A mapping $\phi \colon X \to \mathbb{A}$ is said to be lower semi continuous at x_0 with respect to $\mathbb{A}$ if

$$\|\phi(x_0)\| \leq \lim_{x \to x_0} \inf \|\phi(x)\|.$$

Example 4.3.4.
Let $X = [-1, 1]$ and $\mathbb{A} = \mathbb{R}^2$ be the C^*-algebra with $\|(a_1, a_2)\| = \sqrt{|a_1|^2 + |a_2|^2}$. Define an order $\preceq$ on $\mathbb{A}$ as follows:

$$(x_1, y_1) \preceq (x_2, y_2) \;\Leftrightarrow\; x_1 \leq x_2 \text{ and } y_1 \leq y_,.$$

where "$\leq$" is the usual order on the elements of $\mathbb{R}$. It is easy to see that $\preceq$ is a partial order on $\mathbb{A}_+$. Consider $d \colon X \times X \to \mathbb{A}$ defined by

$$d(x, y) = (|x - y|, 0,)$$

then clearly $(X, \mathbb{A}, d)$ is a C^*-algebra valued metric space. Define a map $\phi\colon X \to \mathbb{A}$ by

$$\phi(x) = \begin{cases} (\frac{x}{2}, 0) & \text{if } x \geq 0 \\ (1, 0) & \text{otherwise.} \end{cases}$$

It is easy to see that ϕ is lower semi continuous at $x_0 = 0$.

It is straightforward to prove the following lemma.

Lemma 4.3.5.
Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space and let $\phi\colon X \to \mathbb{A}_+$ be a map. Define the order $\preceq_\phi$ on X by

$$x \preceq_\phi y \Longleftrightarrow d(x, y) \preceq \phi(y) - \phi(x) \text{ for any } x, y \in X. \tag{4.8}$$

Then $\preceq_\phi$ is a partial order on X.

Theorem 4.3.6. [91]
Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space and $\phi\colon X \to \mathbb{A}_+$ be a lower semi-continuous map. Then $(X, \preceq_\phi)$ has a minimal element, where $\preceq_\phi$ is defined by (4.8).

Proof. Let $x_1 \succeq_\phi x_2 \succeq_\phi x_3 \succeq_\phi \cdots$ be a non-increasing sequence in X, then from (4.8) we have

$$\theta \preceq d(x_2, x_1) \preceq \phi(x_1) - \phi(x_2), \quad \theta \preceq d(x_3, x_2) \preceq \phi(x_2) - \phi(x_3), \ \cdots .$$

$$\Rightarrow \ \phi(x_1) \succeq \phi(x_2) \succeq \phi(x_3) \succeq \cdots .$$

Hence $\{\phi(x_\alpha) : \alpha \in I\}$ is a decreasing chain in $\mathbb{A}_+$, where I is an indexing set.

Let $\{\alpha_n\}$ be an increasing sequence of elements from the indexing set I such that

$$\lim_{n\to\infty} \phi(x_{\alpha_n}) = \inf\{\phi(x_\alpha) \ : \ \alpha \in I\}. \tag{4.9}$$

Taking $m >, n$ we have $x_{\alpha_n} \succeq_\phi x_{\alpha_m}$. It follows from (4.8) that

$$\begin{aligned} d(x_{\alpha_m}, x_{\alpha_n}) &\preceq \phi(x_{\alpha_n}) - \phi(x_{\alpha_m}) \\ \Rightarrow \|d(x_{\alpha_m}, x_{\alpha_n})\| &\leq \|\phi(x_{\alpha_n}) - \phi(x_{\alpha_m}).\| \end{aligned}$$

Taking $\lim n \to \infty$ together with (4.9), we have

$$\begin{aligned} \lim_{n\to\infty} \|d(x_{\alpha_m}, x_{\alpha_n})\| &\leq \lim_{n\to\infty} \|\phi(x_{\alpha_n}) - \phi(x_{\alpha_m})\| \\ &= \|\inf\{\phi(x_\alpha) \ : \ \alpha \in I\} - \inf\{\phi(x_\alpha) \ : \ \alpha \in I\}\| \\ &= 0. \end{aligned}$$

Therefore $\{x_{\alpha_n}\}$ is a Cauchy sequence in X. As X is complete, there exists $x \in X$ such that $x_{\alpha_n} \to x$. Since $\{x_{\alpha_n}\}$ is a decreasing chain in X, it follows that $x \preceq_\phi x_{\alpha_n}$ for all $n \geq 1$. This implies that x is a lower bound for $\{x_{\alpha_n}\}_{n\geq 1}$.

We claim that x is a lower bound for $\{x_\alpha\}_{\alpha\in I}$.

Let $\beta \in I$ be such that $x_\beta \preceq_\phi x_{\alpha_n}$ for all $n \geq 1$. Then

$$\theta \preceq d(x_{\alpha_n}, x_\beta) \preceq \phi(x_{\alpha_n}) - \phi(x_\beta).$$

Taking limit $n \to \infty$ implies

$$\phi(x_\beta) \preceq \inf\{\phi(x_\alpha) : \alpha \in I\}. \tag{4.10}$$

Since $\beta \in I$, we have

$$\inf\{\phi(x_\alpha) : \alpha \in I\} \preceq \phi(x_\beta). \tag{4.11}$$

Combining (4.10) and (4.11) we get

$$\inf\{\phi(x_\alpha) : \alpha \in I\} = \phi(x_\beta). \tag{4.12}$$

As $x_\beta \preceq_\phi x_{\alpha_n}$ it follows from (4.8) that

$$d(x_\beta, x_{\alpha_n}) \preceq \phi(x_{\alpha_n}) - \phi(x_\beta).$$

Using (4.12) and the fact that $\{x_{\alpha_n}\}$ is a decreasing chain in X, we get

$$\theta \preceq d(\lim_{n\to\infty} x_{\alpha_n}, x_\beta) \preceq \lim_{n\to\infty} \phi(x_{\alpha_n}) - \phi(x_\beta) = \phi(x_\beta) - \phi(x_\beta) = \theta.$$

Thus

$$d(\lim_{n\to\infty} x_{\alpha_n}, x_\beta) = \theta.$$

Hence $\lim_{n\to\infty} x_{\alpha_n} = x_\beta$. It follows from the uniqueness of limit that $x_\beta = x$, i. e., x is a lower bound of $\{x_\alpha : \alpha \in I\}$. Hence by using Zorn's Lemma we conclude that X has a minimal element. □

As a consequence of the above theorem, we have the following fixed-point result.

Theorem 4.3.7. [91]
Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space and $\phi\colon X \to \mathbb{A}_+$ be a lower semi continuous map. Let $T\colon X \to X$, be such that for all $x \in X$

$$d(x, Tx) \preceq \phi(x) - \phi(Tx). \tag{4.13}$$

Then T has at least one fixed point.

Proof. Let $a \in X$ be a minimal element of X. Since $Ta \in X$, it follows that $a \preceq_\phi x$ for all $x \in X$, in particular

$$a \preceq_\phi Ta. \tag{4.14}$$

Combining (4.14) and the condition (4.13), we have $Ta = a$, that is, T has a fixed point. □

Example 4.3.8.
Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{R}^2$ be a C^*-algebra with the partial order as given in Example 4.3.4. Define $d\colon X \times X \to \mathbb{A}$ by

$$d(x, y) = (|x - y|, 0).$$

Let $\phi\colon X \to \mathbb{A}_+$, $\phi(x) = (x, 0)$ be continuous map, and $T\colon X \to X$ given by the formula $T(x) = x^2$. Then it is easy to see that all the conditions of Theorem 4.3.7 are satisfied and T has a fixed point. Note that contractive theorem stated in [63] is not applicable here, since contractive condition (3.2) does not hold.

Now we are going to improve the Caristi's theroem by taking the Caristi's contractive condition on the edges of the graph. The technique to prove this result is different from that given in Theorem 4.3.7. Moreover the results are illustrated by constructing nontrivial examples. We first give the following definition and a theorem as an aid for establishing the new result.

Definition 4.3.9.
A mapping $T : X \to X$ is G***-edge preserving*** if $\forall\ x, y \in X$,

$$(x, y) \in E(G) \Longrightarrow (Tx, Ty) \in E(G).$$

Definition 4.3.10.
A mapping T is said to be a ***Caristi*** G***-mapping*** if there exists a lower continuous function $\phi : X \to [0, \infty)$ such that

$$d(x, Tx) \leq \phi(x) - \phi(Tx), \text{ whenever } (Tx, x) \in E(G).$$

Theorem 4.3.11. [4] (Caristi's fixed point theorem on a metric space endowed with a graph)
Let G be an oriented graph on the set X with $E(G)$ containing all loops and suppose that there exists a distance d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous edge preserving, and a G-Caristi mapping. Then T has a fixed point if and only if there exist $x_0 \in X$, with $(T(x_0), x_0) \in E(G)$.

Theorem 4.3.12.
Let $(X, \mathbb{A}, d)$ be a complete C^*-valued metric space endowed with the graph $G = (V(G), E(G))$ with $V(G) = X$. Let $T : X \to X$ be an edge preserving mapping and $\phi : X \to \mathbb{A}_+$ be a G-lower semi continuous mapping satisfying the following conditions:

(i) for each $x \in X$, we have

$$d(x, Tx) \preceq \phi(x) - \phi(Tx) \text{ whenever } (x, Tx) \in E(G), \tag{4.15}$$

(ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$

(iii) T is G-continuous.

Then T has a fixed point.

Proof.
By hypothesis (ii) we have $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. Let $Tx_0 = x_1$, then from (i) we get

$$d(x_0, x_1) \preceq \phi(x_0) - \phi(x_1).$$

Since T is edge preserving, $(x_1, Tx_1) \in E(G)$. Let $Tx_1 = x_2$. Continuing in this way, we get a sequence $\{x_n\}$ in X such that $(x_n, x_{n+1}) \in E(G)$ and

$$d(x_n, x_{n+1}) \preceq \phi(x_n) - \phi(x_{n+1}) \quad \text{for each } n \in \mathbb{N}. \tag{4.16}$$

This implies that

$$\phi(x_{n+1}) \preceq \phi(x_n). \tag{4.17}$$

Therefore, $\{\phi(x_n)\}$ is a non increasing sequence in $\mathbb{A}_+$, thus there exists an $a \succeq \theta$, such that

$$\lim_{n\to\infty} \phi(x_n) = a. \tag{4.18}$$

Let $m, n \in \mathbb{N}$ with $m > n$, by using triangular inequality and (4.16), we have

$$d(x_n, x_m) \preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots d(x_{m-1}, x_m)$$

$$\preceq \phi(x_n) - \phi(x_{n+1}) + \phi(x_{n+1}) - \phi(x_{n+2}) + \cdots + \phi(x_{m-1}) - \phi(x_m)$$

$$\preceq \phi(x_n) - \phi(x_m).$$

Since $d(x_n, x_m)$ and $\phi(x_n) - \phi(x_m)$ are positive elements of the C^*-algebra $\mathbb{A}$, it further implies that

$$\|d(x_n, x_m)\| \leq \|\phi(x_n) - \phi(x_m)\|.$$

Now taking into account (4.18) and letting $n \to \infty$, we have

$$\lim_{n,m\to\infty} \|d(x_n, x_m)\| = 0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X and by completeness of X we have $x_n \to x^* \in X$. $G-$continuity of T further implies that

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(x^*).$$

$$\Rightarrow \quad T(x^*) = x^*.$$

□

Example 4.3.13.

Let $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ be the algebra of all 2×2 matrices with real entries as defined in Example (4.1.3). Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{A}$ by

$$d(x, y) = \begin{pmatrix} |x - y| & 0 \\ 0 & |x - y| \end{pmatrix}. \tag{4.19}$$

It is easy to check that $(\mathbb{R}, \mathbb{A}, d)$ is a complete C^*-valued metric space. Define $T : \mathbb{R} \to \mathbb{R}$ by

$$Tx = \frac{x^2}{2},$$

and consider the graph $G = (V(G), E(G))$, where $V(G) = \mathbb{R}$ and

$$E(G) = \Big\{ \Big(\frac{1}{2^n}, \frac{1}{2^{2n+1}}\Big) : n = 1, 2, \cdots \Big\} \cup \Big\{ (x, x) : x \in \mathbb{R} \Big\}. \tag{4.20}$$

Note that, for each $n \in \mathbb{N}$,

$$(T\frac{1}{2^n}, T\frac{1}{2^{2n+1}}) = (\frac{1}{2^{2n+1}}, \frac{1}{2^{4n+3}}) \in E(G).$$

Define $\phi : X \to \mathbb{A}_+$ by

$$\phi(x) = \begin{pmatrix} |x| & 0 \\ 0 & |x| \end{pmatrix}. \tag{4.21}$$

It can be easily seen that

$$d\Big(\frac{1}{2^n}, \frac{1}{2^{2n+1}}\Big) = \begin{pmatrix} \Big|\frac{1}{2^n} - \frac{1}{2^{2n+1}}\Big| & 0 \\ 0 & \Big|\frac{1}{2^n} - \frac{1}{2^{2n+1}}\Big| \end{pmatrix}$$

$$= \phi\Big(\frac{1}{2^n}) - \phi(\frac{1}{2^{2n+1}}\Big).$$

Hence all the conditions of Theorem 4.4.4 are satisfied and as a result 0 is the fixed point. Note that the contractive condition introduced by Ma et al. [63] is not satisfied here, for example, at $x = 0,\ y = 5$.

Example 4.3.14.

Let $X = \mathbb{C}$ and consider the algebra $\mathbb{A} = \mathbb{C}^2$ with pointwise operations of addition, scalar multiplication and multiplication. Note that positive elements for algebra of complex numbers are non negative reals ([66], [51]) and hence

$$\mathbb{A}_+ = \{(z_1, z_2) : z_1,\ z_2 \text{ are non negative reals}\}.$$

Let $G = (V(G), E(G))$ be a graph with $V(G) = X$ and

$$E(G) = \left\{ \left(\frac{1}{2^n}, \frac{1}{2^{2n+1}}\right) : n = 1, 2, \cdots \right\} \cup \left\{(x, x) : x \in \mathbb{R}\right\}. \tag{4.22}$$

Note that $\|w\| = \max(|z_1|, |z_2|)$ where $w = (z_1, z_2)$ defines a norm on $\mathbb{A}$ and $* : \mathbb{A} \to \mathbb{A}$, given by $w^* = (\bar{z}_1, \bar{z}_2)$, defines a convolution on $\mathbb{C}^2$. Further,

$$\begin{aligned}\|ww^*\| &= \|\left(|z_1|^2, |z_2|^2\right)\| \\ &= \max\left(|z_1|^2, |z_2|^2\right) \\ &= (\max(|z_1|, |z_2|)^2 \\ &= \|w\|^2.\end{aligned}$$

Thus $\mathbb{A}$ becomes a C^*-algebra. Define $d : X \times X \to \mathbb{A}_+$ by

$$d(z_1, z_2) = (|z_1 - z_2|, 0).$$

Clearly $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued metric space. Now define a lower semi continuous map $\phi : X \to \mathbb{A}_+$ by

$$\phi(z) = (|z|, 0)$$

and $T : X \to X$ by

$$Tx = \frac{x^2}{2}.$$

For $(x, Tx) = \left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right) \in E(G)$, we have

$$d(x, Tx) = d\left(\frac{1}{2^n}, \frac{1}{2^{2n+1}}\right)$$

$$= \left(\left| \frac{1}{2^n} - \frac{1}{2^{2n+1}} \right|, 0 \right)$$

$$= \left(\left| \frac{1}{2^n}| - |\frac{1}{2^{2n+1}} \right|, 0 \right)$$

$$= \phi \left(\frac{1}{2^n} \right) - \phi \left(\frac{1}{2^{2n+1}} \right)$$

$$= \phi(x) - \phi(Tx).$$

Hence all the conditions of Theorem 4.4.4 are satisfied and as a result T has a fixed point. Note that the contractive condition introduced by Ma et al. [63] is not satisfied here , for example, at $z_1 = 2 + i,\ z_2 = 2 - i$. As

$$d(Tz_1, Tz_2) = \left(\left| \frac{(2+i)^2}{2} - \frac{(2-i)^2}{2} \right|, 0 \right)$$

$$= (4, 0)$$

$$= \left(\sqrt{2}, 0 \right) (2, 0) \left(\sqrt{2}, 0 \right)$$

$$= a^* d(z_1, z_2) a,$$

where $\|a\| = \max(|\sqrt{2}|, 0) > 1$.

Theorem 4.3.15.
Let $(X, \mathbb{A}, d)$ be a complete C^*-valued metric space endowed with the graph $G = (V(G), E(G))$ with $E(G) = X$. Let $T : X \to X$ be an edge preserving map and let $\phi : X \to \mathbb{A}_+$ be a lower semi continuous mapping such that for each $x \in X$ we have,

$$d(Tx, T^2x) \preceq \phi(x) - \phi(Tx) \quad \text{whenever} \quad (x, Tx) \in E(G). \tag{4.23}$$

Assume that

(a) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$,

(b) there exists a function $g : X \to \mathbb{A}_+$ defined as $g(x) = d(x, Tx)$ is G-lower semi continuous.

Then T has a fixed point.

Proof.
By hypothesis (a), we have $x_0 \in X$ such that $(x_0, Tx_0) =\in E(G)$. Let $x_1 = Tx_0$, from (4.23), we have

$$d(Tx_0, T^2x_0) \preceq \phi(x_0) - \phi(Tx_0)$$

$$\Rightarrow d(x_1, Tx_1) \preceq \phi(x_0) - \phi(x_1).$$

Since T is edge preserving, $(x_0, x_1) \in E(G) \Rightarrow (Tx_0, Tx_1) = (x_1, x_2) \in E(G)$ with $Tx_1 = x_2$. Continuing in the same way, we get a sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n \text{ with } (x_n, x_{n+1}) \in E(G) \ \forall \ n \in \mathbb{N}$$

and

$$d(x_n, x_{n+1}) \preceq \phi(x_{n-1}) - \phi(x_n). \tag{4.24}$$

Now working on the same lines as in the proof of Theorem (4.4.4), it follows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, so we have $x_n \to x^* \in X$ with $(x_n, x_{n+1}) \in E(G)$. By hypothesis (b),

$$g(x) = d(x, Tx)$$

is G-lower semi continuous. Thus we have

$$\begin{aligned}
\|d(x^*, Tx^*)\| &= \|g(x^*)\| \\
&\leq \liminf \|g(x_n)\| \\
&= \liminf \|d(x_n, Tx_n)\| \\
&\leq \liminf \|\phi(x_{n-1}) - \phi(x_n)\| = 0.
\end{aligned}$$

Hence $Tx^* = x^*$. □

4.4 Fixed Point Theorems for C^*-valued G-contractive Type Mappings

Motivated by the contractive condition introduced by Hicks and Rhoades [44] we further weaken the condition used by Ma et al. [63] and establish the following fixed point result. Throughout the next discussion we will consider G as a directed graph with X as set of vertices and $E(G) \supseteq \Delta$.

Definition 4.4.1.
Let $(X, \mathbb{A}, d)$ be a C^*-valued metric space endowed with a G. A mapping $T : X \to X$ is G_T-continuous if for each sequence $\{x_n\}$ in X, $(x_n, x_{n+1}) \in E(G)$ and $x_n \to x$, we have $Tx_n \to Tx$ as $n \to \infty$.

Definition 4.4.2.
Let $(X, \mathbb{A}, d)$ be a C^*-valued metric space endowed with a G for which $V(G) = X$. A mapping $\varphi : X \to \mathbb{A}$ is said to be G_T-lower semi continuous with respect to $\mathbb{A}$, if for each sequence $\{T^n x\}$ in X such that $(T^n x, T^{n+1}x) \in E(G)$ and $T^n x \to Tx$ then, we have

$$\|\varphi(Tx)\| \leq \liminf_{n\to\infty} \|\varphi(T^n x)\|.$$

Definition 4.4.3.
Let $(X, \mathbb{A}, d)$ be a C^*-valued metric space endowed with a graph $G = ((V(G), E(G))$ with $V(G) = X$. A self map T on X is said to be C^*-valued G-contractive type mapping if there exists $p \in \mathbb{A}$ such that

$$d(Tx, T^2x) \preceq p^* d(x, Tx)p, \quad \|p\| \leq 1 \text{ for all } x \in X \text{ with } (x, Tx) \in E(G). \tag{4.25}$$

Theorem 4.4.4.
Let $(X, \mathbb{A}, d)$ be a complete C^*-valued metric space endowed with the graph G and $T : X \to X$ be a C^*-valued G-contractive type mapping. Then T has a fixed point if

1. there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$,
2. T is edge preserving,

3. their exists a G_T-lower semi continuous map $\varphi : X \to \mathbb{A}$ defined as $\varphi(x) = d(x, Tx)$ for each $x \in X$ with $(x, Tx) \in E(G)$.

Proof.
By hypothesis (1) we have $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. Since T is edge preserving, therefore $(T^n x_0, T^{n+1}x_0) \in E(G)\ \forall\ n \in \mathbb{N}$. Also T is a C^*-valued G-contractive type mapping, so one can see that

$$\begin{aligned}
d(T^n x_0, T^{n+1}x_0) &\preceq p^* d(T^{n-1}x_0, T^n x_0)p \\
&\preceq (p^*)^2 d(T^{n-2}x_0, T^{n-1}x_0)p^2 \\
&\vdots \qquad \vdots \qquad \vdots \\
&\preceq (p^*)^n d(x_0, Tx_0)p^n \quad \text{for } (x_0, Tx_0) \in E(G).
\end{aligned}$$

Let $T^n x_0$ be a sequence in X such that $(T^n x_0,\ T^{n+1}x_0) \in E(G)$. Choosing $n > m$ implies

$$\begin{aligned}
d(T^{n+1}x_0, T^m x_0) &\preceq d(T^m x_0, T^{m+1}x_0) + d(T^{m+1}x_0, T^{m+2}x_0) + \cdots + d(T^n x_0, T^{n+1}x_0) \\
&\preceq \sum_{m}^{n} (p^*)^k d(x_0, Tx_0)p^k.
\end{aligned}$$

Let us call $d(x_0, Tx_0) = \omega$. Therefore,

$$\begin{aligned}
d(T^{n+1}x_0, T^m x_0) &\preceq \sum_{k=m}^{n} (p^*)^k \omega p^k \\
&= \sum_{k=m}^{n} (p^*)^k \omega^{\frac{1}{2}} \omega^{\frac{1}{2}} p^k \\
&= \sum_{k=m}^{n} \left(\omega^{\frac{1}{2}} p^k\right)^* \left(\omega^{\frac{1}{2}} p^k\right)
\end{aligned}$$

$$= \sum_{k=m}^{n} |\omega^{\frac{1}{2}} p^k|^2$$

$$\preceq \sum_{k=m}^{n} \|\omega^{\frac{1}{2}} p^k\|^2 I$$

$$\preceq \|\omega^{\frac{1}{2}}\| \sum_{k=m}^{n} \|p^k\|^2 I$$

$$= \|\omega^{\frac{1}{2}}\| \sum_{k=m}^{n} \|p^{2k}\| I.$$

Here I is the multiplcative identity of $\mathbb{A}$ and $\|p\| \leq 1$, hence

$$d(T^{n+1}x_0, T^m x_0) \to \theta \text{ as } m \to \infty.$$

follows from the fact that

$$\lim_{m\to\infty} \sum_{k=m}^{n} \|p\|^k = 0,$$

as $\sum_{k=m}^{n} \|p\|^k$ is a geometric series with common ratio $\|p\| < 1$. This implies that $\{T^n x\}$ is a Cauchy sequence in X. Completeness of X shows that $T^n x_0 \to x_0 \in X$. By hypothesis (3)

$$\varphi(x) = d(x, Tx)$$

is G_T-lower semi continuous which implies

$$\|d(x_0, Tx_0)\| = \|\varphi(x_0)\|$$

$$\preceq \liminf \|\varphi(T^n x_0)\|$$

$$= \liminf \|d(T^n x_0, T^{n+1} x_0)\|$$

$$= 0.$$

Thus $\|d(x_0, Tx_0)\| = 0$ implies $d(x_0, Tx_0) = 0 \Rightarrow Tx_0 = x_0$. $\square$

Example 4.4.5.
Let $X = \mathbb{C}$ be the set of all complex numbers and consider the algebra $\mathbb{A} = \mathbb{C}^2$ with pointwise operations of addition, scalar multiplication and multiplication. Note that positive elements for algebra of complex numbers are non-negative reals ([66], [51]) and hence

$$\mathbb{A}_+ = \{(z_1, z_2) : z_1,\ z_2 \text{ are non negative reals}\}.$$

Let $G = (V(G), E(G))$ be a graph with $V(G) = X$ and

$$E(G) = \left\{ \left(\frac{1}{k^n}, \frac{1}{k^{2n+1}} \right) : k > 1,\ n = 1, 2, \cdots \right\} \cup \left\{ (x, x) : x \in \mathbb{C} \right\}. \tag{4.26}$$

Note that

$$\|\tau\| = \max(|z_1|, |z_2|), \quad \text{where} \quad \tau = (z_1, z_2)$$

defines a norm on $\mathbb{A}$ and $* : \mathbb{A} \to \mathbb{A}$, given by

$$\tau^* = (\bar{z_1}, \bar{z_2}),$$

defines a convolution on $\mathbb{C}^2$. Thus $\mathbb{A}$ becomes a C^*-algebra.
Define $d : X \times X \to \mathbb{A}_+$ by

$$d(z_1, z_2) = (|z_1 - z_2|, 0).$$

Clearly $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued metric space. Define a mapping $T : X \to X$ by

$$Tx = \frac{x^2}{k},\ k > 1,$$

then for $\left(\frac{1}{k^n}, \frac{1}{k^{2n+1}} \right) \in E(G)$, we have

$$d\left(T\frac{1}{k^n}, T^2\frac{1}{k^n} \right) = d\left(\frac{1}{k^{2n+1}}, \frac{1}{k^{4n+3}} \right)$$

$$= \left(\left| \frac{1}{k^{2n+1}} - \frac{1}{k^{4n+3}} \right|, 0 \right)$$

$$= \left(\frac{1}{\sqrt{k}}, 0\right)\left(\left|\frac{1}{k^{2n}}\right| - \left|\frac{1}{k^{4n+2}}\right|, 0\right)\left(\frac{1}{\sqrt{k}}, 0\right)$$

$$\preceq \left(\frac{1}{\sqrt{k}}, 0\right)\left(\left|\frac{1}{k^{n}}\right| - \left|\frac{1}{k^{2n+1}}\right|, 0\right)\left(\frac{1}{\sqrt{k}}, 0\right)$$

$$= \left(\frac{1}{\sqrt{k}}, 0\right)\left(\left|\frac{1}{k^{n}} - \frac{1}{k^{2n+1}}\right|, 0\right)\left(\frac{1}{\sqrt{k}}, 0\right)$$

$$= p^* d\left(\frac{1}{k^n}, \frac{1}{k^{2n+1}}\right) p$$

$$= p^* d\left(\frac{1}{k^n}, T\frac{1}{k^{n}}\right) p,$$

where $p = \left(\frac{1}{\sqrt{k}}, 0\right)$ and

$$\|p\| = \max\left\{|\frac{1}{\sqrt{k}}|, 0\right\} \leq 1.$$

Hence T is a C^*-valued G contractive type mapping. Also the map $\varphi : X \to \mathbb{A}$ defined as $\varphi(x) = d(x, Tx)$ for each $x \in X$ with $(x, Tx) \in E(G)$ is G-lower semi continuous because when we take a sequence $\left\{\frac{1}{k^n}\right\} \in X$ such that $\frac{1}{k^n} \to 0$ as $n \to \infty$ we have

$$\varphi(0) = \|d(0, T0)\|$$

$$= \left\|\liminf \varphi\left(\frac{1}{k^n}\right)\right\|$$

$$= \left\|\liminf d\left(\frac{1}{k^n}, \frac{1}{k^{2n+1}}\right)\right\|$$

$$= \left\|\liminf \left(\left|\frac{1}{k^n} - \frac{1}{k^{2n+1}}\right|, 0\right)\right\|$$

$$= 0.$$

Hence all the conditions of Theorem 4.4.4 are satisfied and T has a fixed point. Note that the contractive condition introduced by Ma et al. [63] is not satisfied here, as

$$d(Tz_1, Tz_2) = \left(\left|\frac{(z_1)^2}{k} - \frac{(z_2)^2}{k}\right|, 0\right) = \left(\left|\frac{(z_1 - z_2)(z_1 + z_2)}{k}\right|, 0\right).$$

Choosing $z_1 = 2 + i,\ z_2 = 3 - i$, gives $|z_1 + z_2| = 5$ which implies

$$d\left(Tz_1, Tz_2\right) = \left(\sqrt{\frac{5}{k}}, 0\right) d\left(z_1, z_2\right) \left(\sqrt{\frac{5}{k}}, 0\right) \preceq p^* d(z_1, z_2) p$$

when

$$\|p\| \geq \| \left(\sqrt{\frac{5}{k}}, 0\right) \| > 1, \text{ with } k = 2.$$

4.5 Fixed Point Theorem for C^*-algebra Valued b-metric Space

Bakhtin [10] introduced the noticeable extension of a metric space namely b-metric space. This hunch is later used by Czerwick [29, 30]for establishing certain fixed point results on such abstract metric spaces. Samreen et al. [89] proved some fixed point results on b-metric spaces endowed with graph. Leaping a step ahead Kamran et al. [54] proved a fixed point result on C^*-valued b-metric space. In this section, we establish a fixed point result by giving the notion of G-contraction on b-metric space. we first give some relevant definitions.

Definition 4.5.1. [54]
Let X be a non empty set and $\mathbb{A}$ be a C^*-algebra. A mapping $d\colon X \times X \to \mathbb{A}_+$ is said to be a C^*-algebra valued b-metric on X if the following conditions hold:

(i) $d(x, y) = \theta \Leftrightarrow x = y$ for all $x, y \in \mathbb{A}$,

(ii) d is symmetric, *i.e.* $d(x, y) = d(y, x)$ for all $x, y \in \mathbb{A}$,

(iii) $d(x, y) \preceq \zeta[d(x, z) + d(z, y)]$ where $\zeta \in \mathbb{A}$ be such that $\|\zeta\| \geq 1$ and $x, y, z \in \mathbb{A}$.

The triplet $(X, \mathbb{A}, d)$ is called a C^*-algebra valued b-metric space having the coefficient ζ.

Remark 4.5.2.
From the following arguments, it can be easily deduced that the concept of a C^*-algebra valued b-metric space is more general than the ordinary C^*-algebra valued metric space:

1. By taking $\mathbb{A} = \mathbb{R}$, the new idea of C^*-algebra valued b-metric space becomes the same as discussed in Definition 2.5.1.
2. Definition 4.5.1 becomes the usual C^*-algebra valued metric as defined in [63] by taking $\zeta = I$.

Following example clarifies the fact that there exist some C^*-algebra valued b-metric spaces which are not C^*-algebra valued metric spaces.

Example 4.5.3.
Consider a nonempty set

$$X = \ell_p = \left\{ x_n \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \ \text{ and } \ 0 < p < 1 \right\}.$$

Let $\mathbb{A} = M_2(\mathbb{R})$. For $a = x_n, b = y_n \in \ell_p$, define $d : X \times X \to \mathbb{A}$ as follows:

$$d(a,b) = \begin{pmatrix} \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}} & 0 \\ 0 & \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}} \end{pmatrix}$$

Then by using the following observation in [29], it is easy to show that d is C^*-algebra valued b-metric space but not a usual C^*-algebra valued metric on X.

$$\left(\sum_{n=1}^{\infty} |x_n - z_n|^p \right)^{\frac{1}{p}} \leq 2^{\frac{1}{p}} \left[\left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n - z_n|^p \right)^{\frac{1}{p}} \right]$$

having the coefficient

$$\zeta = \begin{pmatrix} 2^{\frac{1}{p}} & 0 \\ 0 & 2^{\frac{1}{p}} \end{pmatrix} \quad \text{with} \ \ \|\zeta\| = \sqrt{2}\, 2^{\frac{1}{p}}.$$

From now onward, we will use C^*-valued b-metric for C^*-algebra valued b-metric space. We designate $(X, \mathbb{A}, d)$ to a C^*-valued b-metric space.

The following interpretations about C^*-valued b-metric are the trivial assumption from the analogous notions in C^*-valued metric spaces.

Remark 4.5.4. Let $(X, \mathbb{A}, d)$ be a C^*-valued b-metric space, then

(i) A sequence $\{x_n\} \to x$ with respect to the algebra $\mathbb{A}$ if and only if for any $\epsilon >, 0$ there is an $N \in \mathbb{N}$ such that $\|d(x_n, x)\| < \epsilon$ for all $n > N$. Symbolically, we then write $\lim_{n\to\infty} x_n = x$.

(ii) The sequence $\{x_n\}$ is said to be a Cauchy sequence with respect to $\mathbb{A}$ If for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|d(x_n, x_m)\| < \epsilon$ for all $n, m > N$.

(iii) $(X, \mathbb{A}, d)$ is called a complete C^*-valued b-metric space if every Cauchy sequence in X is convergent pertaining to to $\mathbb{A}$.

Definition 4.5.5.
Let $(X, \mathbb{A}, d)$ be a C^*-valued b-metric space and $G = (V(G), E(G)$ be a graph. A mapping $T\colon X \to X$ is said to be G-contraction on X if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx, Ty) \preceq A^* d(x, y)A \quad \text{for every } (x, y) \in E(G). \tag{4.27}$$

Example 4.5.6.
Let $X = [0, \infty)$ and $\mathbb{A} = \mathbb{R}^2$. Define a partial order $\preceq$ on $\mathbb{A}$ given by

$$(x_1, x_2) \preceq (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$

Let $d\colon X \times X \to \mathbb{A}$ be a metric defined as

$$d(x_1, x_2) = ((x_1 - x_2)^2, 0),$$

it is easy to check tht d is a C^*-valued b-metric with $\zeta = (2, 0)$ [54]. Define $T\colon X \to X$ given by

$$Tx = \frac{x}{3} + 7.$$

Then T is a contraction on X with $A = (\frac{1}{3}, 0)$ as shown below:

$$d(Tx_1, Tx_2) \quad = \quad \big((Tx_1 - Tx_2)^2, 0\big)$$

$$= \left(\left(\frac{x_1}{3} - \frac{x_2}{3}\right)^2, 0\right)$$

$$= \left(\frac{1}{3}, 0\right) d(x_1, x_2) \left(\frac{1}{3}, 0\right)$$

Theorem 4.5.7.
Let $(X, \mathbb{A}, d)$ be a C^*-valued b-metric space with coefficient ζ equipped with a graph $G = (V(G), E(G))$. Let $T\colon X \to X$ be a G-contraction with A as contraction constant satisfying $\|\zeta\|\|A\|^2 < 1$. Then T has a fixed point in X if it satisfies the following conditions:

(i) T is edge preserving,

(ii) there exists an $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$,

(iii) for any sequence $\{T^n x\}$ in X such that $T^n x \to y$ with $(T^{n+1}x, T^n x) \in E(G)$, there exiss a subsequence $\{T^{n_k} x\}$ of $\{T^n x\}$ and $n_0 \in \mathbb{N}$ such that $(y, T^{n_k} x) \in E(G)$, $\forall\, k \geq n_0$.
Moreover, if x and y are two fixed points of T and $(x, y) \in E(G)$ then $x = y$

Proof.
$\mathbb{A} = \{\theta\}$ is the trivial case, hence we assume that $\mathbb{A} \neq \{\theta\}$.

From hypothesis (ii), there exists $x_0 \in X$ such that (x_0, Tx_0) is an edge. Since T is edge preserving, therefore

$$(T^{n+1}x_0, T^n x_0) \in E(G),$$

where

$$x_n = T^n x_0 \quad \text{for} \quad n = 1, 2, \ldots.$$

From the contraction condition (4.27) on T, it follows that

$$\begin{aligned} d(T^n x_0, T^{n+1} x_0) &= d(x_n, x_{n+1}) \\ &= d(Tx_{n-1}, Tx_n) \\ &\preceq A^* d(x_{n-1}, x_n) A \end{aligned}$$

$$\begin{aligned}
&= A^*d(Tx_{n-2}, Tx_{n-1})A \\
&\preceq (A^*)^2 d(x_{n-2}, x_{n-1})A^2 \\
&\preceq (A^*)^3 d(x_{n-3}, x_{n-2})A^3 \\
&\preceq (A^*)^n d(x_0, x_1)A^n \\
&= (A^*)^n C A^n,
\end{aligned}$$

where $C = d(x_0, x_1)$. Using triangle inequality **(M3)** and assuming that $m > n$ we have the following

$$\begin{aligned}
d(T^n x_0, T^m x_0) &= d(x_n, x_m) \\
&\preceq \zeta\, d(x_n, x_{n+1}) + \zeta^2 d(x_{n+1}, x_{n+2}) + \cdots + \zeta^{m-n-1} d(x_{m-2}, x_{m-1}) \\
&\quad + \zeta^{m-n-1} d(x_{m-1}, x_m) \\
&\preceq \zeta (A^*)^n C A^n + \zeta^2 (A^*)^{n+1} C A^{n+1} + \cdots + \zeta^{m-n-1} (A^*)^{m-2} C A^{m-2} \\
&\quad + \zeta^{m-n-1} (A^*)^{m-1} C A^{m-1} \\
&= \zeta[(A^*)^n C A^n + \zeta (A^*)^{n+1} C A^{n+1} + \cdots + \zeta^{m-n-2} (A^*)^{m-2} C A^{m-2}] \\
&\quad + \zeta^{m-n-1} (A^*)^{m-1} C A^{m-1} \\
&= \zeta \sum_{k=n}^{m-2} \zeta^{k-n} (A^*)^k C A^k + \zeta^{m-n-1} (A^*)^{m-1} C A^{m-1} \\
&= \zeta \sum_{k=n}^{m-1} \zeta^{k-n} (A^*)^k C^{\frac{1}{2}} C^{\frac{1}{2}} A^k + \zeta^{m-n-1} (A^*)^{m-1} C^{\frac{1}{2}} C^{\frac{1}{2}} A^{m-1}
\end{aligned}$$

$$= \zeta \sum_{k=n}^{m-2} \zeta^{k-n} (C^{\frac{1}{2}} A^k)^* (C^{\frac{1}{2}} A^k) + \zeta^{m-n-1} (C^{\frac{1}{2}} A^{m-1})^* (C^{\frac{1}{2}} A^{m-1})$$

$$= \zeta \sum_{k=n}^{m-2} \zeta^{k-n} |C^{\frac{1}{2}} A^k|^2 + \zeta^{m-n-1} |C^{\frac{1}{2}} A^{m-1}|^2$$

$$\preceq \|\zeta \sum_{k=n}^{m-1} \zeta^{k-n} |C^{\frac{1}{2}} A^k|^2 \| I + \|\zeta^{m-n-1} |C^{\frac{1}{2}} A^{m-1}|^2 \| I$$

$$\preceq \|\zeta\| \sum_{k=n}^{m-2} \|\zeta^{k-n}\| \|C^{\frac{1}{2}}\|^2 \|A^k\|^2 I + \|\zeta^{m-n-1}\| \|C^{\frac{1}{2}}\|^2 \|A^{m-1}\|^2 I$$

$$\preceq \|\zeta\| \sum_{k=n}^{m-2} \|\zeta\|^{k-n} \|C^{\frac{1}{2}}\|^2 \|A^k\|^2 I + \|\zeta\|^{m-n-1} \|C^{\frac{1}{2}}\|^2 \|A^{m-1}\|^2 I$$

$$\preceq \|\zeta\|^{1-n} \|C^{\frac{1}{2}}\|^2 \sum_{k=n}^{m-2} \|\zeta\|^k \|A^2\|^k I + \|\zeta\|^{-n} \|\zeta\|^{m-1} \|C^{\frac{1}{2}}\|^2 \|A^{m-1}\|^2 I$$

$$\preceq \|\zeta\|^{1-n} \|C^{\frac{1}{2}}\|^2 \sum_{k=n}^{m-2} (\|\zeta\| \|A^2\|)^k I + \|\zeta\|^{-n} \|C^{\frac{1}{2}}\|^2 (\|\zeta\| \|A^2\|)^{m-1} I$$

$$\longrightarrow \theta \quad \text{as } m,\, n \to \infty,$$

since $\sum_{k=n}^{m-2} (\|\zeta\| \|A^2\|)^k$ is a geometric series and $\|\zeta\| \|A^2\| < 1$ which follows from the observation that the summation in the first term is a geometric series. Also $\|\zeta\| \|A^2\| < 1$ implies that both

$$(\|\zeta\| \|A^2\|)^{m-1} \to 0$$

and

$$(\|\zeta\| \|A^2\|)^{n-1} \to 0.$$

This implies that $\{T^n x_0\}$ is a Cauchy sequence in X with respect to $\mathbb{A}$ and it follows from the completeness of $(X, \mathbb{A}, d)$ that $T^n x_0 \to x \in X$, *i.e.*

$$\lim_{n\to\infty} T^n x_0 = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T x_{n-1} = x.$$

As $T^n x_0 \to x$ and $(T^{n+1}x_0, T^n x_0) \in E(G)$ for all $n \in \mathbb{N}$, therefore by hypothesis (iii), there exists a subsequence $\{T^{n_k}x\}$ of $\{T^n x\}$ and $n_0 \in \mathbb{N}$ such that $(y, T^{n_k}x) \in E(G)$, $\forall\ k \geq n_0$. Using this and the following fact

$$\begin{aligned}
\theta &\preceq d(Tx, x) \\
&\preceq \zeta[d(Tx, T^{n_k}x_0) + d(T^{n_k}x_0, x)] \\
&\preceq \zeta[d(Tx, Tx_{n_k}) + d(Tx_{n_k}, x)] \\
&\preceq \zeta\ A^* d(x, x_{n_k})A + d(x_{n_k+1}, x) \\
&\longrightarrow \theta \quad \text{as} \quad k \to \infty.
\end{aligned}$$

As a result, we have $Tx = x$.
To prove that x is the unique fixed point, we suppose that $y \in V(G)$ is another fixed point of T such that $(y, x) \in E(G)$. Then again from the contraction condition (4.27), we have

$$\begin{aligned}
\theta &\preceq d(x, y) \\
&= d(Tx, Ty) \\
&\preceq A^* d(x, y)A.
\end{aligned}$$

Taking the norm of $\mathbb{A}$ on both sides, we have

$$\begin{aligned}0 &\leq \|d(x,y)\| \\ &\leq \|A^* d(x,y) A\| \\ &\leq \|A^*\|\|d(x,y)\|\|A\| \\ &= \|A\|^2 \|d(x,y)\|. \\ &\Rightarrow \|d(x,y)\| = 0,\end{aligned}$$

which is only possible when $d(x,y) = \theta$. Hence $x = y$. □

Example 4.5.8.

Let $\mathbb{A} = \mathbb{R}^2$ and $X = [0,\infty)$ induced with a is C^*-valued b-metric de d as in example 4.5.6. Consider a graph $G = (V(G), E(G))$ where $V(G) = X$ and

$$\begin{aligned}E(G) &= \left\{ \left(\frac{1}{2^n}, \frac{1}{2^{2n+1}} \right) : n \in \ \mathbb{N}, x \ \in X \right\} \\ &\cup \left\{ (\frac{1}{2^n}, 0) : n \in \ \mathbb{N}, x \ \in X \right\} \\ &\cup \{(x,x) : x \in X\}.\end{aligned}$$

Define a mapping $T : X \to X$ by $Tx = \dfrac{x^2}{2}$. Now to check the condition 4.27 we consider $(\frac{1}{2^n}, \frac{1}{2^{2n+1}}) \in E(G)$:

$$\begin{aligned}d(T\frac{1}{2^n}, T\frac{1}{2^{n+1}}) &= \left(\frac{1}{2^{2n+1}}, \frac{1}{2^{4n+3}} \right) \\ &= \left((\frac{1}{2^{2n+1}} - \frac{1}{2^{4n+3}})^2, \ 0 \right) \\ &= \frac{1}{4} \left(\left(\frac{1}{2^{2n}} - \frac{1}{2^{4n+2}} \right)^2, 0 \right)\end{aligned}$$

$$\preceq \left(\left(\left(\frac{1}{2^n} - \frac{1}{2^{2n+1}}\right)\left(\frac{1}{2^n} + \frac{1}{2^{2n+1}}\right)\right)^2, 0\right)$$

$$= \left(\frac{1}{2^n} + \frac{1}{2^{2n+1}}, 0\right)\left(\left(\frac{1}{2^n} - \frac{1}{2^{2n+1}}\right)^2, 0\right)\left(\frac{1}{2^n} + \frac{1}{2^{2n+1}}, 0\right)$$

$$= A^* d\left(\frac{1}{2^n}, \frac{1}{2^{2n+1}}\right) A,$$

where we have

$$A = \left(\frac{1}{2^n} + \frac{1}{2^{2n+1}}, 0\right) \quad \text{with} \quad \|A\| \leq 1,$$

and also

$$\|\zeta\| \, \|A\|^2 = \left(\frac{1}{2^{n-1}} + \frac{1}{2^{2n}}\right) < 1$$

which implies that T of Example 4.5.6 fulfills all assumption of Theorem 4.5.7 and also $x = 0$ is a unique fixed point of T. Note that T is not C^*-valued contraction, when $x = 0, \quad y = 7$.

Conclusion

In this thesis, we have proved some fixed point theorems in the settings of C^*-algebras endowed with a graph. These results are the extensions of previous results presented by Ma et al.

It is worth mentioning that the notion of C^*-algebra valued metric space is different from cone metric space [92] as follows:

"Let E be a real Banach space. A cone P in E defines a partial ordering in E as follows: let $x, y \in E$ we say $x \preceq y$ if $y - x \in P$. Using this partial ordering Huang and Zhang [45] introduced the notion of a cone metric space. A cone metric on a nonempty set X is a mapping $d_c : X \times X \to E$ satisfying:
(i) $d_c(x, y) > 0$ for all $x, y \in X$ and $d_c(x, y) = 0$ if and only if $x = y$;
(ii) $d_c(x, y) = d_c(y, x)$ for all $x, y \in X$;

(iii) $d_c(x,z) \leq d_c(x,y) + d_c(y,z)$ for all $x,y,z \in X$.
In fact this notion is not new and was initially defined by Kantorovich [56] as a K-metric space [48, 56]. Huang and Zhang [45] called a mapping $f : X \to X$ a cone contraction if it satisfies following condition.

$$d_c(fx, fy) \leq k d_c(x,y) \;\forall\; x,y \in X \text{ for some } k \in (0,1). \tag{4.28}$$

Then they generalized the Banach contraction principle in the context of cone metric spaces [45, Theorem 1]. Note that $d_c(x,y)$ is an element of the Banach space E and the right hand side of (4.28) is defined, since E is a real Banach space. The set of positive elements in a C^*-algebra forms a positive cone in the C^*-algebra but the underlying vector space is not a real vector space, in general. Therefore, the notion of a C^*-valued metric space seems general than the notion of cone metric space. For example if we consider the set $\mathbb{A}$ of all 2×2 matrices having entries from complex numbers, then $\mathbb{A}$ is a vector space over the field of complex numbers. Also, $\mathbb{A}$ is a C^*-algebra with Euclidean norm. A mapping $T : X \to X$ is said to be a C^*-valued contraction mapping on X, by Ma et al., (Definition 3.1.1) if there exist an A in a C^*-algebra $\mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx, Ty) \preceq A^* d(x,y) A, \quad \text{for all } x,y \in X. \tag{4.29}$$

Observe that the right hand side of (4.29) is defined because $\mathbb{A}$ is an algebra, not necessarily real. Also, observe that, it is not necessary that one can define an involution "$*$" on a normed space. Thus it seems to be difficult that the inequality (4.29) can be reduce to the inequality (4.28). Further note that the proof of the main result by Ma et al. [63] depends on machinery of C^*-algebras. Thus we conclude that the main results of Ma et al. and ours may not follow from the corresponding results of cone metric spaces."

Recently, Kadelburg and Radenovic [50] and Alsulami et al. [6] have written a note on the article by Ma et al. [63]. They presented an other way to obtain the results of Ma et al. from the corresponding results on complete metric spaces. We observe that the authors in [6, 50] used the same results from C^*-algebras to reach at their conclusions which are used by Ma et al. Therefore, we noticed that both approaches are almost the same. The only difference seems to us is that Ma et al. have given the complete and detailed proofs whereas the authors in [6, 50] used the existing results to give brief proofs of their results with the same underlying idea as used by Ma et al. [63]. The results established in this thesis follow the techniques used by Ma et al.

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