CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Fixed Point Theorems in Operator-Valued Metric Spaces

by

Samina Batul

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Dedicated To My (late) Parents

Abstract

Recently, Ma *et al.* have introduced the notions of a C^*-algebra valued metric space and C^*-algebra value contractive mappings. In this dissertation we generalize this new notion of C^*-valued contractive mappings by weakening their introduced contractive conditions in the setting of C^*-algebra valued metric spaces. Using the new notion of C^*-valued contractive type mappings, we establish some fixed point theorems for such mappings. Our result generalizes the result by Ma *et al.* and those contained therein except for the uniqueness. We provide an existence result for an integral equation as an application of C^*-valued contractive type mappings on complete C^*-valued metric spaces. Moreover, in the setting of C^*-algebra valued b-metric spaces, we generalize the Banach contraction principle and establish a fixed point result for a C^*-algebra valued complete b-metric spaces. Finally, for the multivalued mappings, the thesis also introduces the concept of a C^*-algebra valued metric defined on sets and then extends the result of Nadler in this setting.

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First of all thanks to Almighty ALLAH for giving me the strength and courage to complete my work and the I would like to use this space to say a big 'Thank You' to many people who have helped and encouraged me through out the long and difficult process of completing my doctoral studies. At the top of the list is the name of my supervisor, Associate Professor Dr. Tayyab Kamran, whose encouragement, guidance and support from the initial to the final level enabled me to develop an understanding of the subject. It has been an honour to be his PhD student. I appreciate all his contributions of time, ideas, and suggestions to make my Ph.D. experience productive and stimulating.

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Thank you.
Samina Batul
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“By and large it is uniformly true that in mathematics there is a time lapse between a mathematical discovery and the moment it becomes useful; and that this lapse can be anything from 30 to 100 years, in some cases even more; and that the whole system seems to function without any direction, without any reference to usefulness, and without any desire to do things which are useful.”

-John von Neumann

Chapter 1

Introduction

In different areas of mathematics and applied sciences, the problem of the existence of the solution of many mathematical models is equivalent to the existence of a *fixed point problem* for a certain map. The study of fixed points is, therefore, has a central role in many disciplines of applied sciences. The most essential and key part of the theory of fixed points is the existence of the solution of operator equations satisfying certain conditions, for example, Fredholm integral equations, Voltera integral equations, two point boundary value problems in differential equations as well as some eigenvalue problems. A beautiful blend of analysis, topology and geometry has laid down the foundation of the theory of fixed points.

Given a nonempty set X and a self map $T : X \to X$ defined on X, then by a fixed point problem we mean: Is there any element $x \in X$ such that $Tx = x$? Is such an element (if exist) unique?

The solution to this problem depends not only on the properties of the mapping T but also on the structure of the set X on which this map is defined. Many researchers explored the structure of the set X and the properties of the self map T to find the answer to a fixed point problem. One prominent answer to this problem appeared in its abstract form in the PhD thesis of a Polish mathematician Stefan Banach [10] in 1922. His result [10] is known as "Banach Fixed Point Theorem that states conditions which are sufficient for the existence of a fixed point and its uniqueness. More precisely, the Banach fixed point theorem states that if (X, d) is a complete metric space and $T : X \to X$ is a contraction on X, that is there exists a constant $0 \leq \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) \; \forall \; x, y \in X \tag{1.1}$$

then T has a unique fixed point $x \in X$".

As the Condition (1.1) concerns with the contraction mappings, the Banach fixed point theorem is also recognized as the *Banach Contraction Principle* (BCP) in the theory of fixed points. Over the last decades, the BCP turns out to be a very important tool used for the existence of solution of many non linear problems arising in physics and engineering sciences. Together with the answer to the existence of a unique fixed point, the BCP also provides a simple and an efficient algorithm for finding that fixed point.

Since then, many researchers have established the theory of fixed points particularly in two major directions. One by stating the conditions on the mapping T and second, taking the set X as a more general structure.

The first generalization in this direction is the contractive condition of M. Edelstein [35] in which Banach condition (1.1) has been relaxed by taking distinct points from the space X and permitting constant $\alpha = 1$. Later, Rakotch [86] introduced a contractive condition, in which the constant α of contraction condition (1.1) is replaced by a monotonic decreasing function $\alpha : [0, \infty) \rightarrow [0, 1]$. That is,

$$\text{“}d(Tx, Ty) \leq \alpha(t) d(x, y) \text{ for all } \; x, y \in X\text{”}. \tag{1.2}$$

For further such cosiderations on the contractive condtions, we refer to the work presented in [13, 18–20, 23, 35, 80, 86] etc. A comprehensive comparison of these contraction mappings is given in [92].

Because every contraction is continuous. It is natural to ask are there contraction conditions which does not imply the continuity of the mappings.
The first answer of this question was given by Kannan [55] in 1968, in which he has replaced the contraction condition with

$$\text{“}d(T(x), T(y)) \leq a\{d(x, Tx) + d(y, Ty)\} \text{ where } 0 < a < 1/2\text{”}, \tag{1.3}$$

and with some other conditions. Following Kannan, Chatterjea [21] proved a fixed point theorem for operators which satisfy the condition: "there exists $0 < b < 1$

such that

$$d(Tx, Ty) \leq b[d(x, Ty) + d(y, Tx)], \forall\, x, y \in X." \tag{1.4}$$

For the second class of generalizations where the researchers considered the structure of the space on which T is defined. We remark the work of pseudo-metric space [7], metric-like spaces [6], 2-metric spaces [43, 73, 91], partially ordered metric space [65, 66, 78, 79], the cone metric spaces [2, 50], the b-metric spaces [9, 27].

The work of Nadler [71] and Markin [63] in the late sixties extended the theory of fixed points from singlevalued to multivalued mappings. Recall that by a multivalued mapping we mean a mapping T defiend from a nonempty set X to some collection of nonempty subsets of X. That is, Tx is some subset of X. In this case, a point x is called a fixed point of the mapping T if x belongs to the set Tx. For the case of multivalued mappings the existence of fixed points has been studied by many authors under different conditions see for example [24, 25]. Nadler's Theorem has been extended and generalized by many authors by weaking the contractive nature of the mapping under consideration, but with some additional restriction, as for example to take compact valued mappings see for example [25, 40, 68, 95].

In 1905, M. Frećhet1934 [41, 42] introduced the concept of metric spaces. D. Kurepa [61] one of his PhD student introduced more abstract metric space, in which metric takes values in an ordered vector space. In literature, the metric spaces with vector valued metric are known by different names like generalized metric spaces, cone valued metric spaces, vector valued metric spaces and cone metric spaces. Starting from 2007, many authors worked on cone metric spaces over Banach spaces and the existence of fixed points over such spaces see for example [50, 82, 100].

Recently, Ma et al. [62] introduced the notion of C^*-algebra valued metric spaces. They proved certain fixed point theorems, by giving the definition of C^*-algebra valued contractive mapping analogous to Banach contraction principle (1.1). They have also used the concept of expansive mappings and established related fixed point theorems.

In this dissertation we develop a detailed study of fixed points of C^*-valued contractive mappings. In the study a more general notion of C^*-valued contractive type mapping is introduced and a fixed point result is obtained. Our result generalizes the results given in [62]. We have also introduced the notion of a C^*-valued b-metric space and extended the result of [27] in the setting of C^*-algebra.

Inspired by the work of Nadler for the multivalued mappings, we have also introduced the concept of C^*-valued metric defined on sets. In this new setting, a fixed point theorem for C^*-multivalued contraction is established. Our result generalizes the result of Nadler [71].

The rest of the thesis is divided into four chapters that are organized as follows:

In **Chapter 2** we focus on the basic notions and results which are used in the subsequent chapters to present our contribution. In the first two sections, we collected some tools from analysis and the theory of C^*-algebra. Metric spaces and some of its generalizations are considered in the the third section. Contraction mappings and their examples are given in the fourth section whereas fifth section is devoted to present the Banach contraction theorem and some other contractions. The chapter is concluded with the basic notion from the fixed point theory of multivalued mappings.

In **Chapter 3** of the thesis, we first introduce the notion of continuity in the context of C^*-valued metric spaces and show that a C^*-valued contraction map is continuous with respect to our notion of continuity. Then we introduce a C^*-valued contractive type condition and establish a fixed point theorem analogous to the results presented in [48] in the setting of C^*- algebra. We also show that a C^*-valued contractive type map need not be continuous in context of C^*-valued metric. The results of this chapter are published in the following journal paper:

"*Samina Batul and Tayyab Kamran, C^*-valued Contractive Type Mappings, Fixed Point Theory and Applications, (2015), 2015:142.*"

In **Chapter 4** we introduced the idea of C^*-algebra valued b-metric spaces and generalize the result of [27]. As an application of our result we establish an existence result for an integral equation in a C^*-algebra valued b-metric spaces. The material presented in this chapter is published in the following journal article [53]:

"*Kamran et. al., The Banach contraction principle in C^*-algebra valued b metric spaces with application, Fixed Point Theory and Applications, (2016), 2016:10.*"

The notions of bounded sets, closed sets with respect to a C^*-algebra are introduced in **Chapter 5**. These notion are used to define the idea C^*-algebra valued Hausdorff metric on a nonempty set X. The definition of C^*-multivalued contraction mapping is also presented in this chapter. We have proved that a C^*-multivalued contraction mapping on a complete C^*-algebra valued metric space has a fixed point under some conditions. In the setting of C^*-algebra, our result generalizes the result proved by Nadler [71]. All the results of this chapter are submitted to a journal for a possible publication.

Chapter 2

Preliminaries

The objective of this chapter is to present the basic ideas, definitions , examples and results that will be used in the subsequent chapters. In the first two section, we recollect some basic tools and definitions from analysis and the theory of C^*-algebra. In the third section we discuss some generalizations of the idea of a metric space. The Lipschitizian mappings are explained in the forth section. The fifth section is devoted to the famous Banach contraction principal [10] and some related fixed point theorems. In the last section we consider set-valued or multivalued mappings and presents the contractions and some prominent fixed-point theorems for such mappings.

2.1 Some Tools From Analysis

Since we will be using the structure of the objects in metric spaces, in this section we briefly describe and give examples of some of the important ideas from analysis. For the purpose of making comparison between the elements of a nonempty set, we begin with:

Definition 2.1.1. (**Partial Order**)
Let S be a non-empty set. The relation "$\preceq$" is said to be a partial order on S if the following statements are satisfied for each $a, b, c \in S$:

1. $a \preceq a$, (reflexive)

2. $a \preceq b$ and $b \preceq a \Leftrightarrow a = b$, (antisymmetric)

3. $a \preceq b$ and $b \preceq c \Rightarrow a \preceq c$. (transitive)

The set S equipped with a partial order is then called a partially ordered set. Further, if S is a partially ordered set with the partial order "$\preceq$" and if $a, b \in S$ are such that either $a \preceq b$ or $b \preceq a$, then we say that "a and b are comparable" elements of S. If all the elements of a set S are comparable under a partial order "$\preceq$", then S is called a totally ordered set with respect to the order '$\preceq$'.

Example 2.1.2.

1. The set of real numbers $\mathbb{R}$ is a totally ordered set with respect to the usual ordering "$\leq$"of the real numbers.

2. Let $\mathcal{P}(X)$ be the power set of a given nonempty set X and let the relation $\preceq$ be given by the inclusion. That is, for $A, B \in \mathcal{P}(X)$,

$$A \preceq B \text{ means } A \subset B.$$

   Then the relation $\preceq$ is a partial order and $\mathcal{P}(X)$ is a partially ordered set.

Example 2.1.3.
Let $X = \mathbb{R}^2$, and for (x_1, x_2), and (y_1, y_2) in $\mathbb{R}^2$ define and order '$\preceq$' by

$$(x_1, x_2) \preceq (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \quad \text{and} \quad x_2 \leq y_2,$$

where "$\leq$" is the usual order on the elements of $\mathbb{R}$. Then it is easy to see that $\preceq$ is a partial order on $X = \mathbb{R}^2$ and $\mathbb{R}^2$ is a partially ordered set.

Once the elements of a set are comparable, we can then talk about the ideas like lower bounds, upper bounds, infimum, and supremum.

Definition 2.1.4. [58] (**Limit Supremum and Limit Infimum**)
Let $f : X \to \mathbb{R}$ be a real valued function and X be a nonempty subset of $\mathbb{R}$. Then the limit supremum and the limit infimum of f for $\epsilon > 0$ are defined respectively as follows:

$$\limsup_{y \to x} f(y) = \begin{cases} \sup\{f(y) : |x - y| < \epsilon\}, & \text{if the supremum exists} \\ \infty & \text{otherwise.} \end{cases}$$

$$\liminf_{y\to x} f(y) = \begin{cases} \inf\{f(y) : |x-y| < \epsilon\}, & \text{if the infimum exists} \\ -\infty & \text{otherwise.} \end{cases}$$

Keeping in mind the concepts of right and left continuity from calculus, we define more weaker concepts of upper and lower semi-continuity of real-valued functions as follows:

Definition 2.1.5. [58] (**Upper and Lower Semi-continuity**)
"Let (X, d) be a metric space and $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ an *extended* real real-valued function. Then f is said to be upper semi-continuous at x_0 if for every $\epsilon > 0$, $\exists$ a $\delta > 0$ such that

$$\limsup_{x\to x_0} f(x) \leq f(x_0) \text{ whenever } d(x, x_0) < \delta.$$

That is, $f(x) \leq f(x_0) + \epsilon$ for all $x \in B(x_0, \delta)$ when $f(x) > -\infty$, and $f(x) \to -\infty$ as $x \to x_0$ when $f(x_0) = -\infty$. The function f is called upper semi-continuous if it is upper semi continuous at every point of its domain.

Similarly, we say that f is lower semi-continuous at x_0 if for every $\epsilon > 0$ $\exists$ a $\delta > 0$ such that

$$\liminf_{x\to x_0} f(x) \geq f(x_0) \text{ whenever } d(x, x_0) < \delta.$$

The function f is called lower semi-continuous if it is lower semi-continuous at every point of its domain."

If f is upper semi continuous at x, then the images of points do not exceed fx "too much", near x under f, while there is no constraint on how far these images can fall below fx. Similarly, if f is lower semi-continuous at x, then the images of points do not fall below fx, "too much" near x, under f, but they can still be very greater than fx. A function may be lower or upper semi-continuous without being either left or right continuous as illustrated by the following examples.

Example 2.1.6.
The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 2x - x^2 + \dfrac{1}{2} & \text{if } x < 1, \\ 2 & \text{if } x = 1 \\ \dfrac{1}{2} & \text{if } x > 1. \end{cases}$$

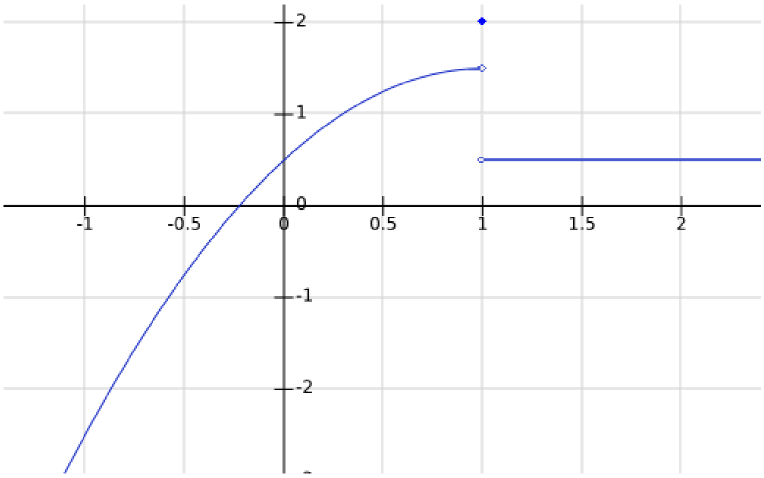

FIGURE 2.1: f is upper semi-continuous at $x = 1$ but not both left as well as right continuous.

is upper semi-continuous at $x = 1$ but not left as well as right continuous.

Example 2.1.7.

Consider another function $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} -1 & \text{if } x = 0, \\ \sin(\dfrac{1}{x}) & \text{if } x \neq 0. \end{cases}$$

Clearly both the left-hand and right hand limits of the function f do not exist at $x = 0$ and hence the function is not left as well as right continuous at $x = 0$. But $\liminf_{x \to 0} f(x) \geq f(0) = -1$ implies that f is lower semi-continuous at $x = 0$.

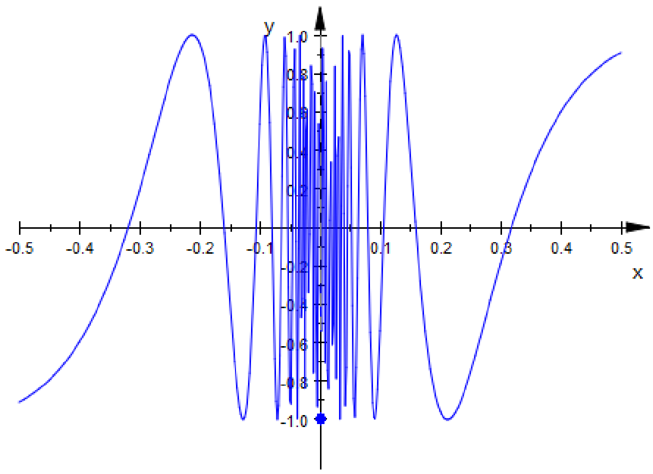

FIGURE 2.2: f is lower semi-continuous at $x = 0$, both left-hand & right-hand limits do not exist at $x = 0$.

Definition 2.1.8. (**Orbit**)
Given a mapping $T : X \to X$ and $x \in X$, the orbit of x with respect to T is defined as the following sequence of points:

$$\mathcal{O}_T(x) = \{x, Tx, T^2x, \dots\}.$$

Example 2.1.9.
Take $X = [-1, 1] \times [-1, 1]$ and define $T : X \to X$ by

$$Tx = T(x_1, x_2) = \begin{cases} (\frac{x_1}{2}, \frac{x_2}{2}) & \text{if } x_1,\, x_2 \geq 0 \\ (1, 0) & \text{otherwise} . \end{cases}$$

Clearly T is not continuous at $(0, 0) \in X$. Taking $x = (x_1, x_2) \in X$ such that $0 < x_1, x_2 < 1$, we have

$$\mathcal{O}_T(x) = \{x, \frac{x}{2}, \frac{x}{4}, \cdots\}.$$

Definition 2.1.10. [48] (**T-orbitally lower semi-continuous**)
A function G from X into the set of real numbers $\mathbb{R}$ is said to be T-orbitally lower semi-continuous at $x' \in X$ if the sequence

$$x_n \subset \mathcal{O}_T(x) \text{ is such that } x_n \to x',$$

we have

$$G(x') \leq \liminf G(x_n).$$

Example 2.1.11. Let $X = [-1, 1]$ and $T : X \to X$ be given by

$$Tx = \frac{x}{2}.$$

Fix x_0 in $(0, 1)$, then the orbit of x_0 with respect to T is given by:

$$\mathcal{O}_T(x_0) = \{x_0, \frac{x_0}{2}, \frac{x_0^2}{4}, \dots\}.$$

Let $\{x_n\}$ be any sequence in $\mathcal{O}_T(x_0)$ then clearly $x_n \to 0$. Consider a function $G : X \to \mathbb{R}$ given by

$$G(x) = |x|.$$

Now $G(0) = 0$ and $x_n \to 0$ implies that

$$\liminf G(x_n) = 0 = G(0).$$

Hence G is T-orbitally lower semi-continuous at $x = 0$.

Definition 2.1.12. (**Fixed Point**)
A fixed point of a mapping $T : X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself, that is,

$$Tx = x.$$

Example 2.1.13.

1. The mapping $x \mapsto x^2$ of $\mathbb{R}$ into itself has two fixed points $x = 0$ and $x = 1$.

2. A translation has no fixed point.

3. If $y = f(x)$ is a real valued function, then, geometrically, the fixed points of f are precisely the points of intersection of the graph of $f(x)$ and the line $y = x$. This is illustrated in the following figures:

The points of intersection of the curve

$$y = f(x) = 2 - x^2 \text{ and the line } \ y = x$$

in Figure 2.3 are the fixed points of f.

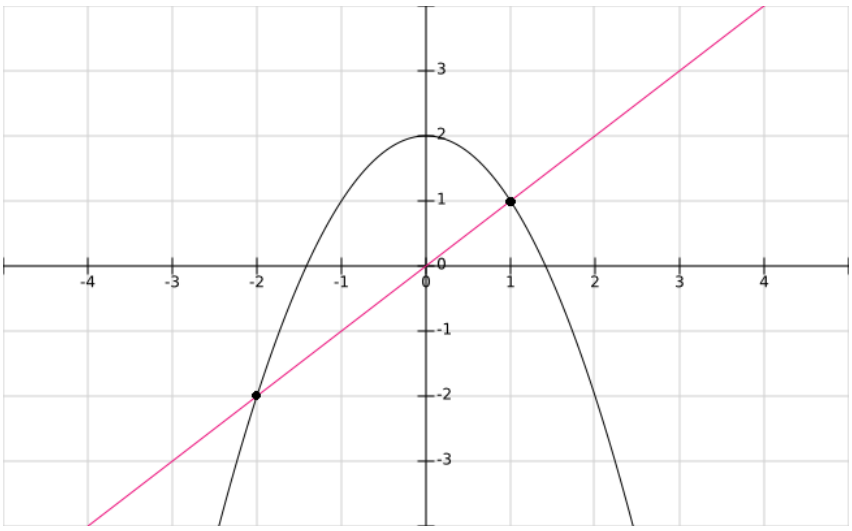

FIGURE 2.3: The fixed points of f are the points of intersection.

Below in Figure 2.4, the curve

$$y = f(x) = \ln(x + \frac{1}{2})$$

does not intersect with the line $y = x$, so

$$f(x) = \ln(x + \frac{1}{2})$$

has no fixed points.

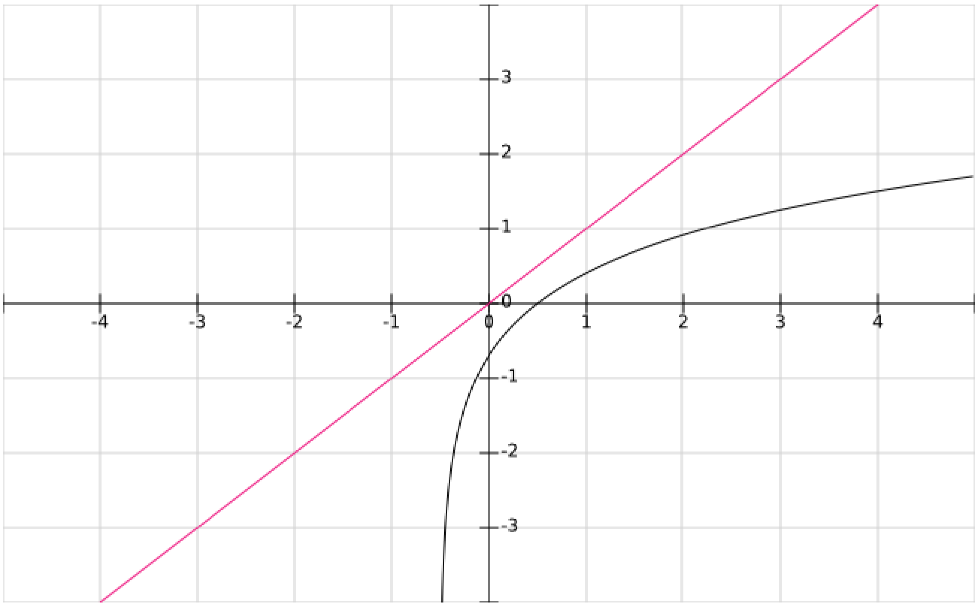

FIGURE 2.4: The function f has no fixed points

Definition 2.1.14. (**multivalued Maps**)
A mapping $T : X \to P(Y)$ is said to be a multivalued, if for each element $x \in X$, Tx is a nonempty subset of Y. In other words, a multivalued map T from a set X to $P(Y)$ is a nonempty subset of the product set $X \times P(Y)$. That is, if $T \subset X \times Y$ is a nonempty set, then T is said to be a multivalued map and the image of an element $x \in X$ under T is denoted by Tx and define by

$$Tx = \{y \in Y | (x, y) \in T\} \subset Y.$$

where X and Y are nonempty sets. The set Tx may be closed, compact, open, bounded, etc.

The inverses of hyperbolic, trigonometric, exponential, integer power functions are all multivalued.

Example 2.1.15.
The inverse of a single valued continuous function $f : X \to Y$ from X onto Y is a

multivalued map $\Psi_f : Y \to X$ defined by

$$\Psi_f(y) = f^{-1}(y) = \{x \in X \ : \ f(x) = y, \quad \text{for} \ \ y \in Y\}.$$

Example 2.1.16.
Take $X = [0, 1]$ and consider
$N(X) = \{A \subset X : A \neq \emptyset\}$
Define $T : X \to N(X)$ and $S : X \to N(X)$ by:

$$\begin{aligned} Tx &= [0, x], \quad \text{and} \\ Sx &= \begin{cases} [0, 1] & \text{if} \ \ x \neq \frac{1}{2}; \\ [\frac{1}{2}, 1] & \text{if} \ \ x = \frac{1}{2}. \end{cases} \end{aligned}$$

Both T and S are multivalued mappings and their graphs are given in Figure 2.5.

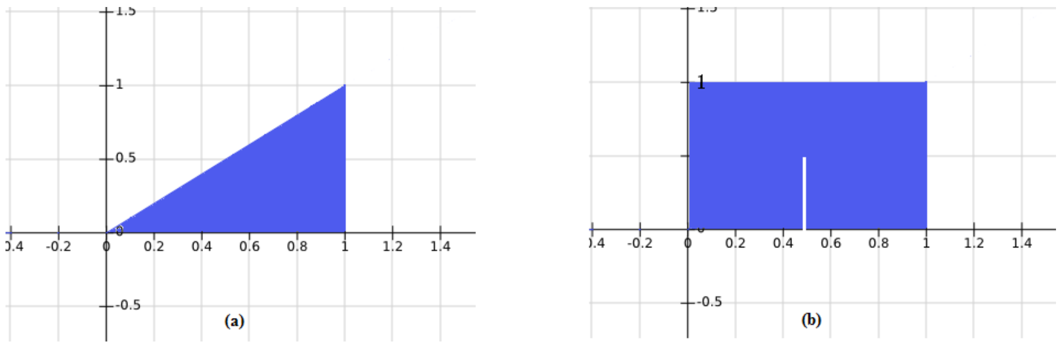

FIGURE 2.5: Graphs of multivalued mappings (a) T and (b) S

Example 2.1.17.
Let $a, b \in \mathbb{R}$ be such that $b > a$.
Define $T : [a, b] \to [a, b]$, by

$$Tx = \begin{cases} [x, b] & \text{if} \ \ a < x < b; \\ \{a, b\} & \text{if} \ \ x \in \{a, b\}. \end{cases}$$

Then T is a multivalued map.

Example 2.1.18.
Let $a, b \in \mathbb{R}$ be such that $b > a$ Define $T : [a, b] \to [a, b]$, by

$$Tx = \{x\} \ \ \text{for all} \ x \in [a, b].$$

Then T is a multivalued map.

2.2 Some Notions from C^*-Algebras

Let $\mathbb{A}$ be vector space over the field of complex numbers $\mathbb{C}$.

Definition 2.2.1. (**Algebra**)

An algebra over $\mathbb{C}$ is a vector space $\mathbb{A}$ with product $(a, b) \mapsto ab$ for each $(a, b) \in \mathbb{A} \times \mathbb{A}$ such that

- $a(bc) = (ab)c$ for all $a, b, c \in \mathbb{A}$, (associativity)
- $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in \mathbb{A}$, (distributivity)
- $(\alpha a)(\beta b) = (\alpha\beta)(ab)$ for all $a, b \in \mathbb{A}$ and $\alpha, \beta \in \mathbb{C}$(compatibility with scalar multiplication).

Definition 2.2.2. (**Normed Algebra**)

A normed algebra is an algebra $\mathbb{A}$ with a norm $\|\cdot\| : \mathbb{A} \to \mathbb{R}$ given by $a \mapsto \|a\|$ such that $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathbb{A}$.

Definition 2.2.3. (**$*$-algebra**)

An algebra $\mathbb{A}$ over $\mathbb{C}$ is called a $*$-algebra if there is an involution map $* : \mathbb{A} \to \mathbb{A}$ satisfying the following conditions

(i) $(a + b)^* = a^* + b^*$ for all $a, b \in \mathbb{A}$,

(ii) $(ca)^* = \bar{c}a^*$ for all $c \in \mathbb{C}$ and for all $a \in \mathbb{A}$,

(iii) $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{A}$

(iv) $(a^*)^* = a$ for all $a \in \mathbb{A}$.

Moreover, if $\mathbb{A}$ contains the identity element 1_A then the pair $(\mathbb{A}, *)$ is called a unital $*$-algebra.

Throughout $1_{\mathbb{A}}$ will denote the multiplicative identity element of $\mathbb{A}$ and $0_{\mathbb{A}}$ is used for the zero element (additive identity) of $\mathbb{A}$.

Definition 2.2.4. (**Banach $*$-algebra**)

A unital $*$-algebra $(\mathbb{A}, *)$ is called a Banach $*$-algebra if

1. $\mathbb{A}$ is a normed algebra *i.e.* $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathbb{A}$,

2. it is complete with respect to this norm.

Definition 2.2.5.
A Banach $*$-algebra $(\mathbb{A}, *)$ such that for all $a \in \mathbb{A}$, we have

$$\|a^*a\| = \|a\|^2.$$

is called a C^*-algebra.

Example 2.2.6.
Let H be a Hilbert space and $H \neq \{0\}$. Consider the space of all bounded linear operators on H and denote it by $B(H)$. Then $B(H)$ is a C^*-algebra with the usual adjoint operations. We define addition and subtraction in $B(H)$ as follows:

1. For all $x \in H$ and S, T in $B(H)$, we have

$$(S+T)(x) = S(x) + T(x)$$

2. For all $x \in H$ and S, T in $B(H)$, the multiplication ST is given by,

$$(ST)(x) = S(x)T(x).$$

It can be verified that $B(H)$ meets all the conditions of a normed algebra with norm defined as follows

$$\|T\| = \sup\{\|T(x)\| : \|x\| = 1\}.$$

To prove that this norm is sub multiplicative we use the fact that if T is a bounded linear operator then

$$\|Tx\| \leq \|T\|\|x\| \text{ for all } x \in H.$$

In fact

$$\begin{aligned} \|(ST)(x)\| &= \|S(Tx)\| \\ &\leq \|S\|\|T\|\|x\| \end{aligned}$$

for all $x \in H$.

Hence

$$\begin{aligned}
\|ST\| &= \sup\{\|S(Tx)\| : \|x\| = 1\} \\
&\leq \sup\{\|S\|\|T\|\|x\| : \|x\| = 1\} \\
&\leq \|S\|\|T\|.
\end{aligned}$$

Also $B(H)$ is complete with this norm and

$$\begin{aligned}
\|T^*T\| &= \{\|T^*Tx\| : \|x\| = 1\} \\
&= \{\langle T^*Tx, x\rangle : \|x\| = 1\} \\
&= \{\langle Tx, Tx\rangle : \|x\| = 1\} = \|T\|^2.
\end{aligned}$$

Example 2.2.7.

Let X be a locally compact Hausdorff space and $C_0(X)$ the set of all continuous functions vanishing at infinity. Define the involution map $* : C_0(X) \to C_0(X)$ as follows:

$$f^*(t) = \overline{f(t)} \text{ for all } t \in X$$

Then it is easy to see that $C_0(X)$ is a $*$-algebra. Define norm on $C_0(X)$ by

$$\|f\| = \sup_{t\in X} |f(t)|.$$

This $\|\cdot\|$ is sub-multiplicative. In fact,

$$\begin{aligned}
\|fg\| &= \sup_{t\in X} |f(t)g(t)| \\
&\leq \sup_{t\in X} |f(t)| \sup_{t\in X} |g(t)| \\
&= \|f\|\|g\|.
\end{aligned}$$

Also

$$\begin{aligned}\|f^*f\| &= \sup_{t\in X} |f^*(t)f(t)| \\ &= \sup_{t\in X} |\overline{f(t)}f(t)| \\ &= \sup_{t\in X} |f(t)|^2 \\ &= \|f\|^2.\end{aligned}$$

Hence $C_0(X)$ is a C^*-algebra.

Definition 2.2.8. (**Spectrum**)
The spectrum $\sigma(a)$ of an element $a \in \mathbb{A}$ is the set of all complex numbers λ for which the element $\lambda 1_{\mathbb{A}} - a$ is not invertible in $\mathbb{A}$. That is,

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda 1_{\mathbb{A}} - a \text{ is non invertible}\}.$$

Example 2.2.9.
Let $\mathbb{A} = M_n$ be the set of all $n \times n$ matrices with complex entries. Then $\mathbb{A}$ is a C^*-algebra with the usual addition and multiplication of matrices. Define the involution $* : \mathbb{A} \to \mathbb{A}$ by

$$A^* = \overline{A}.$$

Then $\mathbb{A}$ is a C^*-algebra and the spectrum of an element $A \in \mathbb{A}$ is the set:

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I_n - A \text{ is non invertible}\},$$

where I_n is the identity matrix of order n. Clearly, the spectrum $\sigma(A)$ is the set of all eigenvalues of A.

The notion of positive elements of a C^*-algebra is defined as follows:

Definition 2.2.10. (**Positive Elements**)
An element $a \in \mathbb{A}$ is called a positive element if $a = a^*$ and $\sigma(a) \subset \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers. If $a \in \mathbb{A}$ is positive, we write it as $a \succeq 0_{\mathbb{A}}$.

We denote by $\mathbb{A}_+$, the set of all positive element of $\mathbb{A}$. That is,

$$\mathbb{A}_+ = \{a \in \mathbb{A} : a \succeq 0_{\mathbb{A}}\}.$$

Positive elements play an important role in C^*- algebras. They determine an order $\succeq$ on self-adjoint elements of $\mathbb{A}$ by

$$b \succeq a \text{ if and only if } b - a \succeq 0_{\mathbb{A}}. \tag{2.1}$$

Example 2.2.11.
Consider again the algebra $\mathbb{A} = M_n$ of Example 2.2.9. Then the positive elements of $A \in \mathbb{A}$ are those matrices A whose eigenvalues are positive real numbers.

We begin with some basic properties of the positive elements.

Lemma 2.2.12. [31]
Each positive element a of a C^*-algebra $\mathbb{A}$ has a unique positive square root.

The following lemma contains some useful characterization of positivity.

Lemma 2.2.13. [31]
If $a = a^*$ in a C^*- Algebra $\mathbb{A}$ then the following conditions are equivalent:

(i) $a \succeq 0$;

(ii) $a = b^2$ for some $b = b^*$;

(iii) $\| c1_{\mathbb{A}} - a \| \preceq c$; for all $c \geq \| a \|$;

(iv) $\| c1_{\mathbb{A}} - a \| \preceq c$ for some $c \geq \| a \|$.

Corollary 2.2.14. [31]
If a and b are positive elements of a C^*-algebra $\mathbb{A}$ then $a + b$ is positive.

Theorem 2.2.15. [31]
If a is an element of a C^*- algebra A, then a^*a is positive.

2.3 Some Generalizations of a Metric Space

Recall that the concept of the“distance” between the points in Euclidean spaces allows us to define a more general concept of distance between two points of an arbitrary nonempty set X that became known as a metric defined on that set X. In deed, "a metric on a nonempty set X is a non-negative real valued map $d : X \times X \to \mathbb{R}^+$ such that for all x, y, z in X, we have **(1)** $d(x, x) = 0$, **(2)** $d(x, y) = d(y, x)$, and **(3)** $d(x, z) \leq d(x, y) + d(y, z)$. The set X together with a metric d defined on it is called a metric space." Now a natural question arises how this general concept of distance, that is, metric d can be further generalized or modified? Can we drop any of the above three conditions that a metric must satisfy? Can we replace any of theses conditions by some weaker condition to define a new concept of a metric? Is it possible to replace the domain of the metric d with some other structure? Fortunately, the answers to these questions is affirmative. In fact, these questions have given rise to the concepts of pseudo-metric space [7], metric-like spaces [6], 2-metric spaces [43, 73, 91], partially ordered metric space [65, 66, 78, 79], the cone metric spaces [2, 50], the b-metric spaces [9, 27], the C^*-algebra valued metric spaces [62] etc.

In this section, we present briefly the ideas of b-mtric spaces introduced by Bakhtin [9] and the concept of a C^*-valued metric spaces [62].

2.3.1 b-Metric Spaces

One of the well known generalization of the idea of a metric space is introduced by Bakhtin [9] and afterward used by Czerwick [27, 28]. They introduced and used the concept of b-metric space to establish certain fixed point results for generalizing Banach contraction principle. The same idea of relaxing the triangular inequality has also been discussed by Fagin et al. [39], who used to call this new distance as nonlinear elastic matching. The idea is clearly an extension of the metric space as follows from the following definition.

Definition 2.3.1 ([59]). Let X be a non-empty set and $b \in \mathbb{R}$ such that $b \geq 1$. A b-metric on X is a real-valued mapping $d_b \colon X \times X \to \mathbb{R}$ that satisfies the following conditions for all $x, y, z \in X$:

1. $d_b(x, y) \geq 0$ and $d_b(x, y) = 0 \Leftrightarrow x = y$;
2. $d_b(y, x) = d(x, y)$;
3. $d_b(y, z) \leq b\,[d_b(y, x) + d_b(x, z)]$.

The pair (X, d_b) is said to be a b metric space. If we take $b = 1$ in this definition, then this definition coincides with the usual definition of a metric space.

Later Q. Xia [101] use this concept to study the optimal transport path between probability measures. Xia has chosen to call these spaces quasimetric spaces, which is the term used in the book by Heinonen [47].

Example 2.3.2. [12, 28]
Consider $X = \ell_p(\mathbb{R})$ where $0 < p < 1$ that is

$$\ell_p(\mathbb{R}) = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}.$$

Define $d_b : X \times X \to \mathbb{R}^+$ as:

$$d_b(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}},$$

where $x = \{x_n\}$, $y = \{y_n\}$ are the elements of ℓ_p. Then (X, d_b) is a b-metric space with coefficient $b = 2^{\frac{1}{p}}$.

Example 2.3.3. [16]
The space $X = C[0, 1]$ of all real valued functions $x(t)$ where t is in the interval $[0, 1]$ such that

$$\int_0^1 |x(t)|^p < \infty$$

is a b-metric space by taking

$$d_b(x, y) = \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{\dfrac{1}{p}},$$

for each $x, y \in C[0, 1]$.

Remark 2.3.4.
It is clear that the usual metric space is a b-metric space . However, Czerwick [27, 28] has shown that a b-metric on X need not be a metric on X.

The following example is an illustration of the above remark.

Example 2.3.5. [8]
Let $X = \{0, 1, 2\}$ and define a metric d_b on X in the following way

$d_b(2,0) = d_b(0,2) = m \geq 2$
$d_b(1,0) = d_b(0,1) = d_b(1,2) = d_b(2,1) = 1$ and
$d_b(0,0) = d_b(1,1) = d_b(2,2) = 0$. Then,

$$d_b(x,y) \leq \frac{m}{2}[d_b(x,z) + d_b(z,y)].$$

for all $x, y, z \in X$. If $m > 2$, then X is not a metric space because it does not satisfy the triangle inequality.

Example 2.3.6. [101]
"Consider a metric space (X, d). For some $\beta > 1, \alpha \geq 0$, $a > 0$, and for $x, y \in X$ define $D : X \to \mathbb{R}$ by

$$D(x,y) = \alpha d(x,y) + a d(x,y)^{\beta}.$$

Then, in general, D is not a metric on X. On the other hand, one can verify that (X, D) is a b-metric space with $b = 2^{\beta-1}$. In fact, let $z \in X$ be arbitrary, then

$$\begin{aligned} D(x,y) &= \alpha d(x,y) + a d(x,y)^{\beta} \\ &\leq \alpha[d(x,z) + d(z,y)] + a[d(x,z) + d(z,y)]^{\beta} \\ &\leq \alpha[d(x,z) + d(z,y)] + 2^{\beta-1} a[d(x,z)^{\beta} + d(z,y)^{\beta}] \\ &\leq 2^{\beta-1}[D(x,z) + D(z,y)]. \end{aligned}$$

The above fact follows from the fact that if a and b are positive real numbers and $\beta > 1$, then

$$\left(\frac{a+b}{2}\right)^{\beta} \leq \frac{a^{\beta} + b^{\beta}}{2}."$$

Subsequently, throughout this section let (X, d_b) be a b-metric space (unless otherwise specified) with co-efficient $b \geq 1$. We recall some auxiliary notions and results in a b-metric space which are needed subsequently.

Definition 2.3.7.
Consider a b-metric space (X, d_b) and $A \subseteq X$ then $\bar{A}$ the closure of A is the set of all limits points of A and points of A.

If
$$A = \bar{A},$$
then A is said to be closed.

Definition 2.3.8.
A sequence $\{x_n\}$ in (X, d_b) is called convergent if and only if there exists $x \in X$ such that:
$$d_b(x_n, x) \to 0 \text{ as } n \to \infty.$$

Definition 2.3.9.
Consider a b-metric space (X, d_b). A sequence $\{x_n\} \in X$ is said to be a Cauchy sequence if and only if
$$d_b(x_n, x_m) \to 0 \text{ as } n, m \to \infty.$$

Definition 2.3.10.
"A b-metric space (X, d_b) is said to be complete if every Cauchy sequence in X is convergent" with respect to the metric d_b.

Remark 2.3.11.
Let (X, d_b) be b-metric space then:

1. If a sequence is convergent then, it has a unique limit;
2. If a sequence is convergent then, it is a Cauchy sequence;
3. This is proved by [29] that a b-metric d_b is not in general a continuous functional.

The following result is proved by [67] in 2013.

Theorem 2.3.12. [67]
Let (X, d_b) be a complete b-metric space with constant $b \geq 1$ and define the sequence $\{x_n\}_{n=1}^{\infty} \subset X$ by the recursion

$$x_n = Tx_{n-1} = T^n x_0, n = 1, 2, \cdots$$

$T : X \to X$ a contraction with the restrictions $k \in [0, 1)$ and $kb < 1$. Then there exists

$$x^* \in X \text{ such that } x_n \to x^*$$

and x^* is the unique fixed point of T.

In the same paper the author generalized the Kannan [55] and Chatterjea [21] fixed point theorems in the setting of b-metric spaces.

Now we are going to give the b-metric version of Cantor's intersection theorem [17] which can be proved easily in a similar way as the proof of its metric version.

Theorem 2.3.13. [17]
Let (X, d_b) be a complete b-metric space then every nested sequence of closed balls has a non-empty intersection.

For the further details on the theory of fixed points in b-metric spaces we refer to [9, 27, 34, 59, 67].

2.3.2 C^*-valued Metric Spaces

Using the notion of positive elements in $\mathbb{A}$, a C^*-algebra valued metric space is defined in the following way.

Definition 2.3.14. [62]
"Let X be a non-empty set. A mapping $d : X \times X \to \mathbb{A}$ is called a C^*-algebra valued metric on X if the map d satisfies the following conditions:

(i) $0_{\mathbb{A}} \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_{\mathbb{A}} \Leftrightarrow x = y$,

(ii) $d(x, y) = d(y, x)\ \forall\ x, y \in X$,

(iii) $d(x, y) \preceq d(x, z) + d(z, y)\ \ \forall\ x, y, z \in X$.

The triplet $(X, \mathbb{A}, d)$ is called a C^*- algebra valued metric space."

Example 2.3.15.
Let $X = [-1, 1]$ and $\mathbb{A} = \mathbb{R}^2$, then $\mathbb{A}$ is a C^* algebra with usual norm

$$\|x\| = \sqrt{x_1^2 + x_2^2}.$$

Further a partial ordering on $\mathbb{A}$ as in Example 2.1.3 is given by

$$(a, b) \preceq (c, d) \Leftrightarrow a \leq c \text{ and } b \leq d,$$

where "$\leq$" is the usual order on the elements of $\mathbb{R}$.
Define $d : X \times X \to \mathbb{A}$ by

$$d(x, y) = (|x - y|, 0),$$

then d is a C^*- algebra valued metric and $(X, \mathbb{A}, d)$ is a C^* algebra valued metric space.

The following definitions are due to Ma et al.

Definition 2.3.16. [62]
"Consider a C^*-algebra valued metric space $(X, \mathbb{A}, d)$ and let $x \in X$.

1. A sequence $\{x_n\}$ in $(X, \mathbb{A}, d)$ is said to be convergent with respect to $\mathbb{A}$, if for any $\epsilon > 0$ there exist a positive integer N such that

$$\|d(x_n, x)\| \leqslant \epsilon \text{ for all } n > N.$$

2. (Cauchy Sequence) A sequence $\{x_n\}$ is called a Cauchy sequence with respect to $\mathbb{A}$ if for any $\epsilon > 0$ there exist a positive integer N such that

$$\|d(x_n, x_m)\| \leqslant \epsilon$$

for all $n, m > N$.

3. If every Cauchy sequence with respect to $\mathbb{A}$ in X is convergent then $(X, \mathbb{A}, d)$ is called a complete C^*-algebra valued metric space."

2.4 Contraction Mappings

Before we state the pivotal fixed point theorem in the theory of functional analysis, we present the following classification of mappings defined on a metric space X.

Definition 2.4.1. (Lipschitzian)
Let $X = (X, d)$ be a metric space. A mapping $T : X \to X$ is called a Lipschitzian if there is a positive real number α such that for all $x_1, x_2 \in X$,

$$d(Tx_1, Tx_2) \leq \alpha d(x_1, x_2). \tag{2.2}$$

The constant α is called Lipschitz constant of the mapping T.

Example 2.4.2.
Let $X = \mathbb{R}$ with the usual metric. Define $T : X \to X$ by $Tx = 2x$, then

$$\begin{aligned} d(Tx_1, Tx_2) &= d(2x_1, 2x_2) \\ &= |2x_1 - 2x_2| \\ &= 2|x_1 - x_2| \\ &= 2d(x_1, x_2), \end{aligned}$$

shows that T is Lipschtizian on X with $\alpha = 2$.

Definition 2.4.3. (Contraction)
Let $X = (X, d)$ be a metric space. " A mapping $T : X \to X$ is called a contraction on X if there is a positive real number $\alpha < 1$ such that for all $x_1, x_2 \in X$".

$$d(Tx_1, Tx_2) \leq \alpha d(x_1, x_2). \tag{2.3}$$

Example 2.4.4.
Consider again the set $X = \mathbb{R}$ of real numbers with the standard metric given by

$$d(x_1, x_2) = |x_1 - x_2| \text{ for all } x_1, x_2 \in X.$$

Let T be a self map on X defined by

$$Tx = \frac{x}{5} + 2,$$

then it is straightforward to see that T is a contraction on X.

Definition 2.4.5. (Contractive)
Consider a metric space $X = (X, d)$. A mapping on X is said to be contractive on X if for each element $x_1, x_2 \in X$,

$$d(Tx_1, Tx_2) < d(x_1, x_2), \qquad x_1 \neq x_2. \tag{2.4}$$

Example 2.4.6.
Let(X, d) be a metric space with $X = [1, \infty)$ and d be the usual metric on X. Define $T : X \to X$, by

$$Tx = x + \frac{1}{x} \;\; \forall \;\; x \in X.$$

Then for all $x_1 \neq x_2 \in X$, we have

$$\begin{aligned} d(Tx_1, Tx_2) &= \mid x_1 + \frac{1}{x_1} - x_2 - \frac{1}{x_2} \mid \\ &= \mid x_1 - x_2 + \frac{1}{x_1} - \frac{1}{x_2} \mid \\ &= \mid x_1 - x_2 + \frac{(x_2 - x_1)}{x_1 x_2} \mid \\ &= \mid x_1 - x_2 \mid\mid 1 - \frac{1}{x_1 x_2} \mid \\ &< \mid x_1 - x_2 \mid . \end{aligned}$$

Thus T is contractive mapping but, as we can see that T is not a contraction.

Definition 2.4.7. (Non-expansive)
A self mapping on a metric space X is called non-expansive on X if for all x_1, x_2 in X we have

$$d(Tx_1, Tx_2) \leq d(x_1, x_2). \tag{2.5}$$

Example 2.4.8.
Let $X = \mathbb{R}$ and d be the usual metric. Let $T = I$ be the identity map on X, that is,

$$Tx = Ix = x,$$

then T is non-expansive map but it is not contractive map on X.

We have following implications:

Contraction $\Rightarrow$ **Contractive** $\Rightarrow$ **Non-expansive** $\Rightarrow$ **Lipschtizian**.

In the setting of C^*-algebra the following definitions and result are due to Ma et al. [62].

Definition 2.4.9. [62] "Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space. A mapping $T : X \to X$ is said to be a C^*-valued contractive mapping on X if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx, Ty) \preceq A^* d(x, y) A, \quad \text{for all } x, y \in X.\text{"} \tag{2.6}$$

2.5 Some Generalization of Banach Contraction Principle

We start with the most celebrated result of the theory of metric fixed points, that is, the Banach fixed point theorem also known as the Banach contraction principle. Despite of its simplicity, it is the most widely used applied fixed point theorem. In fact, it is an an existence and uniqueness theorem for fixed points of mappings that are contractions and requires only the completeness of the underlying metric space. It, not only, establishes the existence of fixed points but also give a constructive algorithm for obtaining better and better approximations to that fixed point. That is, it establishes an iterative scheme for finding the fixed point of such mappings. Moreover, the proof of Banach theorem has become an important tool for proving many fixed point results in the literature.

Theorem 2.5.1. Banach Fixed Point Theorem
Every contraction on a complete metric space (X, d) has a unique fixed point. More precisely, "let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \text{for all } x, y \in X$$

and for some $\alpha \in [0, 1)$. Then T has exactly one fixed point x_0 and for every $x \in X$", the sequence of points

$$\{T^n x\} = \{x, Tx, T^2 x, \ldots\} \tag{2.7}$$

converges to this fixed point x_0.

The above theorem not only gives an algorithm to find an approximation of the fixed point of a contraction but also gives error bounds as stated in the following corollary.

Corollary 2.5.2. (Error Bounds)[60]
"Suppose the mapping T satisfies all the conditions of Theorem 2.5.1. Then the iterative sequence (2.7) with arbitrary $x \in X$ converges to the unique fixed point x of T. The following inequalities give the prior and the posterior error bounds:

$$d(x_m, x) \leq \left(\frac{\alpha^m}{1-\alpha}\right) d(x_0, x_1). \tag{2.8}$$

$$d(x_m, x) \leq \left(\frac{\alpha}{1-\alpha}\right) d(x_{m-1}, x_m).\text{''} \tag{2.9}$$

The following theorem given by Picard-Lindel illuminates the validity of Banach fixed point theorem.

Theorem 2.5.3. Picard Iteration Theorem[26].
Consider the following initial value problem

$$x'(t) = f(t, \ x(t)), \quad x(t_0) = x_0.$$

Suppose the following assertions hold:

1) f is continuous within and on a rectangular region

$$R = \{(t, x) : |t - t_0| \leq a, |x - x_0| \leq b\}.$$

2) f satisfies the Lipschitz condition with respect to x, or equivalently

$$|f(t, x_1) - f(t, x_2)| \leq c|x_1 - x_2|, \text{ for } (t, x_1), (t, x_2) \in R \text{ and } c > 0.$$

Then the above initial value problem has a unique solution $x(t)$ with t in the interval $[t_0 - \beta, t_0 + \beta]$ and $\beta < \min\{a, \frac{b}{k}, \frac{1}{c}\}$, where by the boundedness of f we have

$$|f(t, x)| \leq k.$$

The significance of Banach principle can be seen in its lot of applications in different areas such as the existence of equilibria in game theory, existence of solutions to differential equations etc. This is one of the most important motivation for the researchers to work to originate a collection of generalizations of Theorem 2.5.1. This work resulted in many interesting fixed point theorems by introducing new forms of contraction conditions that involving not only $d(x, y)$, the distance between x and y but also the the distances between their images fx and fy under changing of x and y under the mapping f i.e., $d(x, fx), d(y, fy), d(x, fy), d(y, fx)$.

2.5.1 Some Other Contractions

The present section is devoted to present some weaker form of contractions. For instance, Edelstein [35] defined the notion of contractive mappings see Definition 2.4.9. In order to get a fixed point of a map that is contractive, we have to add additional assumption like the space is compact or $\exists$ an $x \in X$ for which $\{f^n(x)\}$ contains a convergent subsequence.
To establish a fixed point result for a non-expansive map we also need to enforce some conditions such as the space must be compact or some other assumptions.
A self mapping T on X is called weakly contractive if

$$d(Tx_1, Tx_2) \leq d(x_1, x_2) - \Psi(d(x_1, x_2)) \text{ for all } x_1, x_2 \in X,$$

Here the map Ψ is a self map defined on $[0, \infty]$ which is continuous and non decreasing such that $\Psi > 0$ on $(0, \infty)$, $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \infty$. By taking the underlying space as a Hilbert space, Alber and Guerre-Delabriere [5] introduced the notion of "weakly contractive mapping". In fact, they showed that "each weakly contractive mapping defined on a Hilbert space have a unique fixed point, without any extra assumption". Afterwards Rhoades [93] proved the validity of this result for metric spaces. From the definition it is clear that the "weakly contractive maps" are sandwiched between contraction and contractive maps.

In 1969 Kannan [56] proved three interesting theorems for the existence of fixed points. In the first theorem, he had ommitted the completeness of the space and obtained the same conclusion as in Banach's Theorem but with different sufficient conditions. He states his result as follows:

Theorem 2.5.4. [56]**Kannan Fixed Point Theorem**
"Let E be a metric space with the metric d and let T be a map of E into itself such that

1. $d(T(p), T(q)) \leq \alpha\{d(p, T(p)] + d[q, T(q))\}$, where $0 < \alpha < \dfrac{1}{2}$ and $p, q \in E$.
2. T is continuous at a point x_0 of E.
3. There exists a point $x \in E$ such that the sequence of iterates $T^n(x)$ has a sub sequence $T_i^n(x)$ converges to x_0,

then x_0 is the unique fixed point of T".

In [55] he has proved the following theorem:

Theorem 2.5.5. Kannan Fixed Point Theorem
"If T is a map of a complete metric space X into itself and if for all $x, y \in X$ we have

$$d(T(x), T(y)) \leq \alpha\{d(x, T(x)) + d(y, T(y))\}, \text{ where } 0 < \alpha < \frac{1}{2}, \tag{2.10}$$

then T has a unique fixed point".

Example 2.5.6.
Let $X = [0, 1]$ with the usual metric. Define $T : X \to X$ by

$$T(x) = \begin{cases} 1 - x & \text{if } x \text{ is irrational in } [0,1] \\ \dfrac{1+x}{3} & \text{if } x \text{ is rational in } [0,1], \end{cases}$$

is a Kannan map on $[0, 1]$ and has $x_0 = \dfrac{1}{2}$ as a fixed point. Also it is continuous at its fixed point.

A mapping T satisfying (2.10) need not be continuous in the whole domain. The Kannan fixed point theorem [55] played an important role in the development of fixed point theory of generalized contractive mappings and was soon followed by a large number of interesting papers on contractive mappings. The Kannan fixed point theorem also gave rise to the famous question of continuity of contractive mappings at their fixed points. The question of the existence of contractive mappings which are discontinuous at their fixed points was settled by Pant [80, 81].

In [21] Chatterjea proved that "if a mapping $T : Y \to Y$ of a complete metric space (Y, d) with the metric d satisfies the condition

$$d(Tx, Ty) \leq \beta[d(x, Ty) + d(y, Tx)],\ 0 \leq \beta < \frac{1}{2}, \tag{2.11}$$

for all $x, y \in Y$, then T has a unique fixed point in Y". Such a mapping T need not be continuous in the entire domain.

Remark 2.5.7. All of the above contractive conditions are independent of each other. As we can see from the following examples.

Example 2.5.8.
Let $X = [0, 1]$ and let d be the usual metric on $\mathbb{R}$. Define $T : X \to X$ as follows:

$$Tx = \frac{3}{4}x, \quad \text{for all} \ \ x \in X.$$

Then T is the Banach contraction but does not satisfy Kannan condition (2.10). For example if $x = 0, y = 1$ then

$$d(Tx, Ty) = \frac{3}{4},\ d(x, Tx) + d(y, Ty) = 0 + \frac{1}{4} = \frac{1}{4}$$

and hence

$$d(Tx, Ty) > d(x, Tx) + d(y, Ty).$$

Example 2.5.9.
Let $X = [0, 1]$ and d be the usual metric. Define $T : X \to X$ defined by

$$T(x) = \begin{cases} \frac{x}{4} & \text{if } 0 \leq x < 1 \\ \frac{1}{3} \text{ if} & x = 1. \end{cases}$$

Then $d(Tx, Ty) < \alpha[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$ for each $\beta \in [0, 1/2)$. Therefore T satisfy the Chatterjea's contraction condition (2.11) but T does not satisfies the Banach contraction condition.

Example 2.5.10.
Let $X = [-1, 1]$ equipped with the usual metric on $\mathbb{R}$ and define $T : X \to X$ by

$$Tx = -\frac{3}{4}x$$

for all $x \in X$. Then T satisfies the Banach contraction condition but does not satisfies the Chatterjea's contraction condition (2.11). For example if $x = -1,\ y = 1$, then

$$d(Tx, Ty) = \frac{3}{2},\ d(x, Ty) + d(y, Tx) = \frac{1}{2}$$

and hence

$$d(Tx, Ty) > d(x, Ty) + d(y, Tx).$$

Theorem 2.5.11.
Let X be a metric space with d as metric and let T be a map of X into itself. Suppose that T is continuous at a point $x_0 \in X$. If there exists a point $x \in E$ such that the sequence of iterates $T^n(x)$ converges to x_0 then $Tx_0 = x_0$. If in addition

$$d(Tx_0, Tx) \leq \alpha d(x, x_0),\ x_0 \in X, 0 < \alpha < 1,$$

then x_0 is the unique fixed point of T.

The above notion of Kannan's contraction was more refined by Ciric [23], Rus [94], Reich [90], Hardy and Rogers [46] and also by Zamfirescu [102]. "The function $f : X \to X$ is called Ciric- Reich-Rus operator if there exists $a, b, c \in \mathbb{R}$ and $a, b, c \geq 0$ with $a + b + c < 1$ such that

$$d(fx, fy) \leq ad(x, y) + bd(x, fx) + cd(y, fy) \text{ for all } x, y \in X\text{"}$$

Hardy and Rogers [46] extended the class of Ciric-Reich-Rus operators by considering the following condition:

$$\text{"}d(fx, fy) \leq ad(x, y) + bd(x, fx) + cd(y, fy) + ed(x, fy) + fd(y, fx)$$

for all $x, y \in X$ and $a + b + c + e + f < 1$".
In [93] Rhoades has proved the following theorem

Theorem 2.5.12.
"Consider a complete metric space (X, d) and consider a mapping $f : X \to X$ which is weakly contractive. Then there exists a unique fixed point of f'."

In [35, 74] Nemytzki and Edelstein states the following result:

Theorem 2.5.13.
"Let(X, d) be a compact metric space and $f : X \to X$ be a contractive mapping. Then there exists a unique fixed point of f".

Now we are going to give two "equivalent" facts. The first is a well-known variational principle due to Ekeland [37, 38] and the second is the well-known Caristi Theorem [20].

Theorem 2.5.14. The Caristi-Ekeland Principle [37]
Let (M, d) be a complete metric space and if $\phi : M \to \mathbb{R}^+$ a lower semi-continuous mapping. Define a relation "$\preceq$" as follows

$$x \preceq y \text{ if and only if } d(x, y) \leq \phi(x) - \phi(y), \quad x, y \in M.$$

Then $(M, \preceq)$ has a minimal element.

Carisiti's fixed point theorem is one of the beautiful extensions of Banach's contraction principle. This Theorem states that

Theorem 2.5.15. Caristi [20]
Let (M, d) be a complete metric space.and there exists a lower semi-continuous mapping ϕ from M into the non-negative real numbers and suppose $g : M \to M$ satisfies:

$$d(x, g(x)) \leq \phi(x) - \phi(g(x)), \quad x \in M.$$

Then g has a fixed point.

The following result is proved by Hicks and Rhoades [48] in 1979 and then they showed that many generalization of Banach fixed point theorem 2.5.1 are special cases of their theorems. It states that

Theorem 2.5.16. [48]
"Let $h \in [0, 1)$ be such that

$$d(Ty, T^2y) \leq hd(y, Ty) \text{ for every } y \in \mathcal{O}_T(x). \tag{2.12}$$

where x is a fixed element in X. Then:

(A1) $\exists\, x_0 \in X$ such that the sequence $T^n x$ converges to x_0,

(A2) $d(T^n x, x_0) \leq \dfrac{h^n}{1-h} d(x, Tx)$,

(A3) x_0 is a fixed point of T if and only if $G(x) = d(x, Tx)$ is T-orbitally lower semi continuous at x_0".

The following examples describes the diversity that can happen when a function T satisfies only condition (2.12).

Example 2.5.17. [48]
Let $X = [-1, 1]$ and d be the usual metric on $\mathbb{R}$. Define $T : X \to X$ by

$$T(x) = \begin{cases} \dfrac{x}{4} \text{ if } x \neq 0, \\ \\ \dfrac{1}{4} \text{ if } x = 0. \end{cases}$$

For $x \neq 0$ we have,

$$d(Tx, T^2x) = \frac{1}{4} d(x, Tx)$$

Similarly for $x = 0$,

$$d(Tx, T^2x) = \frac{3}{4} d(x, Tx).$$

But T has no fixed point and $\lim T^n x = 0$ for every x.

Example 2.5.18. [48]
Let $X = [-1, 1]$ and define $T : X \to X$ by

$$T(x) = \begin{cases} \dfrac{x}{4} \text{ if } x > 0, \\ -1 \text{ if } x \leq 0. \end{cases}$$

For each $x > 0$,

$$d(Tx, T^2x) = \frac{1}{4} d(x, Tx).$$

Also for $x > 0$, it easy to see that

$$\lim_{n\to\infty} T^n x = 0,$$

but $x = -1$ is the unique fixed point of T.

2.5.2 Contractions on an Ordered Metric Space

After that Turinici [98], Ran and Reurings [87] fixed points on a metric space of self mappings on which some partial odered is defined are studied and different authors have established many interesting results. For instance, see [1, 30, 44, 76, 85, 87, 89]. Consider a partially ordered set $(X, \preceq)$ and $f : X \to X$. The map f is called non-increasing if for $x_1, x_2 \in X$, we have

$$x_1 \preceq x_2 \Rightarrow f(x_1) \succeq f(x_2).$$

The mapping f is called non-decreasing if for $x_1, x_2 \in X$,

$$x_1 \preceq x_2 \Rightarrow f(x_1) \preceq f(x_2).$$

If we have for $x_1 \preceq x_2$ either $f(x_1) \preceq f(x_2)$ or $f(x_2) \preceq f(x_1)$ for all $x_1, x_2 \in X$ then f maps comparable elements to comparable elements.
Ran and Reuring [87] has proved the following result

Theorem 2.5.19.
"Consider a complete metric space (X, d) and let $\preceq$ be a partial order on X. Let the mapping $f : X \to X$ satisfies the condition

$$d(f(x_1), f(x_2)) \leq kd(x_1, x_2) \text{ for all } x_1, x_2 \in X \text{ and } x_1 \preceq x_2,$$

where $k \in (0, 1)$. Also if the following axioms hold

1. there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$ or $f(x_0) \preceq x_0$;

2. f is monotone and continuous:

3. Every pair of elements of X has an upper and lower bound.

Then f is a Picard operator".

Theorem 2.5.19 is further enhanced by Nieto and Rodriguez-Lopez in [76] further enhanced and as an application, they have also presented an existence result for the solution of ordinary differential equation with periodic boundary value conditions.

Theorem 2.5.20. Nieto and Rodriguez-Lopez [76]
"Consider a complete metric space (X, d) and $\preceq$ is a partial order defined on X. If a self map $f : X \to X$ is non decreasing with respect to the given partial order and meets the condition

$$d(f(x_1), f(x_2)) \leq kd(x_1, x_2) \text{ with } x_1 \preceq x_2, \text{ for all } x_1, x_2 \in X$$

where $k \in (0, 1)$ and suppose that one of the following conditions is also true :

i) f is continuous and there exists $x_0 \in X$ such that either $x_0 \preceq f(x_0)$ or $f(x_0) \preceq x_0$;

ii) for any non decreasing sequence x_n in X if $x_n \to x$ then $x_n \preceq x$ for $n \in \mathbb{N}$ and there exists $x_0 \in X$ such that $(x_0) \preceq f(x_0)$;

iii) for any non increasing sequence x_n in X if $x_n \to x$ then $x_n \succeq x$ for $n \in \mathbb{N}$ and there exists $x_0 \in X$ such that $x_0 \succeq f(x_0)$.

Then f has a fixed point. Also, if every pair of elements of X has an upper or a lower bound then f is a Picard operator."

Working in the same direction the authors [75, 77, 89] extended the above results by relaxing the continuity condition as stated in the following theorem.

Theorem 2.5.21.
"Let (X, d) be a complete metric space endowed with a partial order $\preceq$. Let $f : X \to X$ preserves comparable elements and satisfies

$$d(f(x_1), f(x_2)) \leq kd(x_1, x_2) \text{ for all } x_1, x_2 \in X \text{ with } x_1 \preceq x_2,$$

where $k \in (0, 1)$. Suppose the following conditions hold:

i) either f is orbitally continuous; or
   if for any sequence $x_n \to x$ and for each pair of comparable elements (x_n, x_{n+1}) there exist a subsequence x_{n_k} such that the pair of elements (x_{n_k}, x) are comparable for $k \in \mathbb{N}$,

ii) there exists $x_0 \in X$ such that the pair $(x_0, f(x_0))$ is comparable.

Then f has a fixed point. Furthermore, if every pair of elements of X has an upper or a lower bound then f is a Picard operator".

The following theorems give some interesting results of fixed points for self maps that satisfy some contractive or expansive conditions on C^*-valued metric spaces proved by Ma et al. [62].

Theorem 2.5.22. [62]
"If $(X, \mathbb{A}, d)$ is a C^*-algebra valued metric space and T satisfies (2.6), then there exists a unique fixed point in X."

Like the idea of contractive mappings, another important concept is the concept of expansive mappings as defined below:

Definition 2.5.23. [62]
"Let $X \neq \emptyset$. A self mapping T is said to be a C^*-algebra valued expansive mapping, if T satisfies:

i) $T(X) = X$;

ii) $d(Tx, Ty) \succeq A^* d(x, y) A, \forall\, x, y \in X$,

where $A \in \mathbb{A}$ is an invertible element and $\|A^{-1}\| < 1$."

For the expansive mappings, we further have a fixed point theorem.

Theorem 2.5.24.
"Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. Then for the expansive mapping T, there exists a unique fixed point in X".

Before introducing another fixed point theorem, we first state the following lemma from [32, 69].

Lemma 2.5.25.
"Suppose that $\mathbb{A}$ is a unital C^*- algebra with a unit $1_{\mathbb{A}}$.

1. If $a \in \mathbb{A}_+$ with $\| a \| < \frac{1}{2}$, then $1_{\mathbb{A}} - a$ is invertible and $\| a(1_{\mathbb{A}} - a)^{-1} \| < 1$.

2. Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq 0_{\mathbb{A}}$ and $ab = ba$, then $ab \succeq 0_{\mathbb{A}}$.

3. By A' we denote the set $\{a \in \mathbb{A} : ab = ba, \forall\, b \in \mathbb{A}\}$. Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq 0_{\mathbb{A}}$ and $1_{\mathbb{A}} - a \in \mathbb{A}'_+$ is an invertible operator, then $(1_{\mathbb{A}} - a)^{-1}b \succeq (1_{\mathbb{A}} - a)^{-1}c$."

Theorem 2.5.26. [62]
"Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space. Suppose that the mapping $T : X \to X$ satisfies

$$d(Tx, Ty) \preceq A(d(Tx, y) + d(Ty, x)) \text{ for all } x, y \in X$$

where $A \in \mathbb{A}'_+$ and $\|A\| < \dfrac{1}{2}$. Then there exists a unique fixed point of T in X".

From now on, we will call a "C^*-algebra valued metric space" and a "C^*-algebra valued metric", a 'C^*-valued metric space' and a 'C^*-valued metric' respectively.

2.6 Fixed Point Theorems for Multivalued Mappings

In this section we are going to collect some fixed point results of multivalued mappings which are already present in the literature.

The class of spaces which have fixed point property for continuous set valued mappings is small than that class of spaces which have fixed point property for continuous single valued mappings. Such results can be found in [99]. Using the Hausdroff metric the study of fixed points for multivalued contraction and non-expansive mapping was first studied by Markin [63] and Nadler [70, 71]). Afterward an interesting and rich theory for fixed point of multivalued maps was developed which has application in control theory, convex optimazition and economics see [45].

Definition 2.6.1. (**Fixed Points in Maultivalued Maps**)
Consider two nonempty sets X and Y and let $T : X \to P(Y)$ be a multivalued mapping. Then $x \in X$ is called fixed point of T, if $x \in Tx$.

Example 2.6.2.
Let $X = [0, 1]$. Define $T : X \to N(X)$ by

$$Tx = [0, x^2].$$

Then $0 \in T0 = \{0\}$ and $1 \in T1 = [0, 1]$ and hence both 0 and 1 are fixed points of T.

Definition 2.6.3. (**Common Fixed Points**)
Let X and $P(Y)$ be two nonempty sets and $T_1, T_2 : X \to P(Y)$ be two multivalued mappings. A point $x \in X$ is called common fixed point of both T_1 and T_2, if $x \in T_1x \cap T_2x$.

Example 2.6.4.
Let $a, b \in \mathbb{R}$ be such that $b > a$.
Define $T_1, T_2 : [a, b] \to P([a, b])$, and by

$$T_1x = \begin{cases} \{a\} & \text{if } \; x = \{a, b\}; \\ [x, b] & \text{if } \; a < x < b, \end{cases}$$

and

$$T_2x = [a, x] \;\forall\; x \in [a, b].$$

Then each $x \in [a, b)$ is a common fixed point of T_1 and T_2.

We denote by $CL(X)$ the collection of all nonempty closed subsets of X, the collection of nonempty bounded closed subsets of X by $CB(X)$, the collection of nonempty compact subsets of X by $K(X)$.

Theorem 2.6.5.
Let X be a nonempty set and d be a meric on X. Let $CB(X)$ be the collection of non-empty closed and bounded subsets of X.

Define the map $H : CB(X) \times CB(X) \to \mathbb{R}$ as follows:

$$H(A, B) = \max\{\sup_{a\in A} d(a, B),\; \sup_{b\in B} d(b, A)\}, \tag{2.13}$$

for each $A, B \in CB(X)$. Then H is a metric on $CB(X)$. Where

$$d(a, B) = \inf\{d(a, b) : b \in B\}.$$

This is called Hausdorff metric on $CB(X)$. This is also known as the Hausdorff distance between the sets in $CB(X)$ generated by the metric d on X. The pair $(CB(X), H)$ is called Hausdorff metric space.

It is well known that the Hausdorff metric space $(CB(X), H)$ is complete when the metric space (X, d) is complete.

The following lemmas are needed in continuation.

Lemma 2.6.6. [49]
Let $A, B \in CB(X)$ and $\epsilon > 0$ with $H(A, B) < \epsilon$, then for each $a \in A$, there exists an element $b \in B$ such that

$$d(a, b) < \epsilon.$$

Lemma 2.6.7. [33]
Let $A, B \in CB(X)$. Then for each $a \in A$

$$d(a, B) \leq H(A, B).$$

In 1969 Nadler [71] combine the idea of multivalued mapping with Lipschitz condition and extended the Banach contraction principle to multivalued maps in the following way.

Theorem 2.6.8. [71]
Let $\alpha \in [0, 1)$ and (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a mapping such that

$$H(Tx, Ty) \leq \alpha d(x, y) \ \forall \ x, y \in X. \tag{2.14}$$

Then T has a fixed point.

Contrary to Banach Theorem 2.5.1, Nadler's Theorem 2.6.8 does not assert the uniqueness of the fixed point. Several researchers then started work to extend and develop the theory of fixed point for multivalued mappings. Nadler fixed point theorem has been extended in 1972 by Reich [90] in the following way.

Theorem 2.6.9. [90]
If (X, d) is a complete metric space and $T : X \to K(X)$ satisfies

$$H(Tx_1, Tx_2) \leq \alpha(d(x_1, x_2))d(x_1, x_2) \tag{2.15}$$

for each $x_1, x_2 \in X$ where $\alpha : [0, \infty) \to [0, 1)$ such that

$$\limsup_{r \to t^+} \alpha(r) < 1 \tag{2.16}$$

for each $t \in (0, \infty)$, then T has a fixed point.

For further results and applications of fixed points of multivalued mappings we refer to [24, 25, 68, 95].

Chapter 3

Generalized C^*-valued Contractions

Let (X, d) be a metric space, $x \in X$ and $T : X \to X$ be a mapping on X. The sequence of points

$$\mathcal{O}_T(x) = \{x, Tx, T^2x, \dots\}$$

is called the orbit of x with respect to T. Hicks and Rhoades [48] showed that if the mapping T satisfies the following contractive condition

$$d(Ty, T^2y) \leq hd(y, Ty) \tag{3.1}$$

for some $h \in (0, 1)$ and every $y \in \mathcal{O}_T(x)$ then T has a fixed point. Note that the contractive condition (3.1) is weaker than the condition (2.5.1). Moreover, the condition (3.1) does not forces that the mapping T to be continuous [48]. In contrast to Banach contraction principle, Hicks and Rhodes theorem [48] does not guarantee the uniqueness of the fixed point of T.

In this chapter, we introduce a new contractive conditions for the mappings on C^*-algebra valued metric spaces and establish related fixed point theorems for self maps with contractive conditions. Our result generalizes Theorem 2.1 (Ma et al.) presented in [62]. We conclude this chapter with an application of our result and establish an existence theorem for an integral equation.

Throughout this chapter, X is a nonempty set and $\mathbb{A}$ is a C^*-algebra.

3.1 Banach Type C^*-valued contractions

In this section, we first introduce the notion of continuity in the context of C^*-algebra valued metric spaces and show that a C^*-valued contraction map is continuous with respect to our notion of continuity. Then we introduce a C^*-valued contractive type condition and establish a fixed point theorem analogous to the results presented in [48]. We also show that a C^*-valued contractive type map need not be continuous in context of C^*-valued metric.

Definition 3.1.1. **(Continuity with respect to** $\mathbb{A}$**)**
Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space. A mapping $T : X \to X$ is said to be continuous at x_0 with respect to $\mathbb{A}$ if given any $\epsilon > 0$ there exist a $\delta > 0$ such that:

$$\|d(Tx, Tx_0)\| \leqslant \epsilon \quad \text{whenever} \quad \|d(x, x_0)\| \leqslant \delta.$$

Further, T is said to be continuous on X with respect to $\mathbb{A}$ if it is continuous for every $x \in X$.

Example 3.1.2.
Let $\mathbb{A} = \mathbb{R}^2$ and $X = [0, 1]$. Then $\mathbb{A}$ is a C^*-algebra with norm $\|\cdot\| : \mathbb{A} \to \mathbb{R}$ defined by

$$\|(x, y)\| = (x^2 + y^2)^{1/2}.$$

Define a C^*-algebra valued metric $d : X \times X \to \mathbb{A}$ on X by

$$d(x, y) = (|x - y|, 0) \tag{3.2}$$

with ordering on $\mathbb{A}$ as given in Example 2.1.3 by

$$(a, b) \preceq (c, d) \Leftrightarrow a \leq c \text{ and } b \leq d. \tag{3.3}$$

A mapping $T : X \to X$ given by

$$T(x) = \frac{x}{3}$$

is continuous with respect to $\mathbb{A}$. since

$$\|d(Tx, Ty)\| = \|d(\frac{x}{3}, \frac{y}{3})\| = \|\frac{x}{3} - \frac{y}{3}\| < \epsilon$$

with $\|x - y\| < 3\epsilon = \delta$.

Remark 3.1.3.

Note that every continuous map is continuous with respect to $\mathbb{A} = \mathbb{R}$.

Definition 3.1.4. (T**-orbitally lower semi-continuous**)

A function $f : X \to \mathbb{A}$ is said to be T-orbitally *lower semi continuous* at x_0 with respect to $\mathbb{A}$ if the sequence $\{x_n\}$ in $\mathcal{O}_T(x)$ is such that $\lim_{n\to\infty} x_n = x_0$ with respect to $\mathbb{A}$ implies

$$\|f(x_0)\| \leqslant \liminf \|f(x_n)\|. \tag{3.4}$$

We illustrate the above definition by the following example.

Example 3.1.5.

Consider the C^*-algebra $\mathbb{A} = \mathbb{R}^2$ as defined in Example 2.3.15.

Let $X = [-1, 1]$ and define $f : X \to \mathbb{A}$ by

$$f(x) = \begin{cases} (\frac{x}{2}, 0) & \text{if } x \geq 0 \\ \\ (|x-1|, 0) & \text{if } x < 0. \end{cases}$$

By taking $T : X \to X$, defined by

$$Tx = \frac{x^2}{2}$$

we see that, for $\dfrac{1}{2} \in [-1, 1]$, we have

$$\mathcal{O}_T\left(\frac{1}{2}\right) = \left\{\frac{1}{2}, \frac{1}{2^3}, \frac{1}{2^7}, \frac{1}{2^{15}}, \cdots\right\}$$

and any sequence $\{x_n\}$ in $\mathcal{O}_T(x)$ converges to 0. Further,

$$\|f(0)\| = \|(0, 0)\| = \liminf \|f(x_n)\|.$$

Thus f is T-orbitally lower semi continuous at $x = 0$.

Remark 3.1.6.

If $\mathbb{A} = \mathbb{R}$ then our definition of T-orbitally lower semi-continuous maps with respect to $\mathbb{A}$ coincides with usual definition of T-orbitally lower semi-continuous as defined in [48].

Example 3.1.7.
Let $X = [-1, 1]$ and $\mathbb{A} = \mathbb{R}^2$ with the C^*-algebra valued metric $d : X \times X \to \mathbb{A}$ given by (see Example 2.3.15)

$$d(x, y) = (|x - y|, 0),$$

and with the ordering $\preceq$ defined on $\mathbb{A}$ be as follows:

$$(a, b) \preceq (c, d) \Leftrightarrow a \leq c \text{ and } b \leq d.$$

Define $T : X \to X$ by

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}$$

Then T is not continuous at 0. For $0 < x_* < 1$, we have

$$\mathcal{O}_T(x_*) = \left\{x_*, \frac{x_*}{2}, \frac{x_*}{4}, \frac{x_*}{8}, \cdots\right\}.$$

Let $\{x_n\}$ be any sequence in $\mathcal{O}_T(x_*)$, then

$$x_n \to 0 \text{ with respect to } \mathbb{A},$$

and define $f : X \to \mathbb{A}$ by

$$f(x) = d(x, Tx),$$

now by (3.4)

$$\begin{aligned} \|f(0)\| &= \|d(0, T0)\| \\ &\leq \liminf \|f(x_n)\| \\ &= \liminf \|d(x_n, Tx_n)\| \\ &= \|(0, 0)\|, \end{aligned}$$

and hence f is lower semi-continuous at 0.

Definition 3.1.8. (**Contractive Type Mapping**)
Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space. A mapping $T : X \to X$ is said to be a C^*-valued contractive type mapping if there exists an $x \in X$ and an $a \in \mathbb{A}$

such that

$$d(Ty, T^2y) \preceq a^* d(y, Ty)a \ \text{ with } \|a\| < 1 \ \text{ for every } y \in \mathcal{O}_T(x). \tag{3.5}$$

Remark 3.1.9.
A C^*-valued contractive map need not be continuous with respect to the C^*-algebra $\mathbb{A}$.

Remark 3.1.10.
A C^*-valued contraction mapping is C^*-valued contractive type mapping but the converse is not true in general.

The following example supports our above claims.

Example 3.1.11.
Let $X = [-1, 1]$ and $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ with

$$\|A\| = \left(\sum_{i=1}^{4} |a_i|^2 \right)^{\frac{1}{2}}$$

where $a_i s$ are the entries of the matrix $A \in M_{2\times 2}(\mathbb{R})$. Then $(X, \mathbb{A}, d)$ is a C^* algebra valued metric space, where

$$d(x, y) = \begin{bmatrix} |x-y| & 0 \\ 0 & |x-y| \end{bmatrix}$$

and partial ordering on $\mathbb{A}$ is given as

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \succeq \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \Leftrightarrow a_i \geq b_i \text{ for } i = 1, 2, 3, 4.$$

Which is the partial order induced by the cone $\mathbb{A}_+$ as follows:

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \succeq \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} - \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in \mathbb{A}_+.$$

Define a mapping $T : X \to X$ by

$$T(x) = \begin{cases} \frac{x}{4} & \text{if } x \geq 0 \\ 1 & \text{if } x < 0. \end{cases}$$

Then for $y \in \mathcal{O}_T(x)$, $x \geq 0$

$$d(Ty, T^2y) = \begin{bmatrix} |\frac{y}{4} - \frac{y}{16}| & 0 \\ 0 & |\frac{y}{4} - \frac{y}{16}| \end{bmatrix}$$

But, $\frac{y}{4} - \frac{y}{16} = \frac{3y}{16}$, it follows that:

$$d(Ty, T^2y) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} |\frac{3y}{4}| & 0 \\ 0 & |\frac{3y}{4}| \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$= a^* d(y, Ty) a$$

where

$$a = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

and so

$$\|a\| = \frac{1}{\sqrt{2}}.$$

Thus T is a C^*-valued contractive type mapping. Note that T is not continuous with respect to the C^*-algebra $\mathbb{A}$ and hence not a C^*-valued contraction mapping.

Before giving our main result we prove the following lemma which is essentially extracted from the proof of Theorem 2.5.22.

Lemma 3.1.12.
Consider a C^*-algebra valued metric space $(X, \mathbb{A}, d)$ and a self mapping $T : X \to X$. If $a \in \mathbb{A}$ is such that $\|a\| < 1$ then for $m < n$ we have

$$\lim_{n\to\infty} \sum_{k=m}^{n} (a^*)^k d(x, Tx) a^k = 1_{\mathbb{A}} \|(d(x, Tx))^{1/2}\|^2 \big(\frac{\|a\|^m}{1 - \|a\|}\big) \tag{3.6}$$

and

$$\sum_{k=m}^{n} (a^*)^k d(x,Tx) a^k \longrightarrow 0_{\mathbb{A}} \quad \text{as } m \longrightarrow \infty. \tag{3.7}$$

Proof.

Since $d(x,Tx)$ is a positive element of $\mathbb{A}$, we have

$$\begin{aligned}
\sum_{k=m}^{n} (a^*)^k d(x,Tx) a^k &= \sum_{k=m}^{n} (a^*)^k (d(x,Tx))^{1/2} (d(x,Tx))^{1/2} a^k \\
&= \sum_{k=m}^{n} ((d(x,Tx))^{1/2} a^k)^* (d(x,Tx))^{1/2} a^k)
\end{aligned}$$

But for positive elements $a \in \mathbb{A}$, we have $a^* a = |a|^2$.

Therefore, it follows that

$$\begin{aligned}
\sum_{k=m}^{n} (a^*)^k d(x,Tx) a^k &= \sum_{k=m}^{n} |(d(x,Tx))^{1/2} a^k|^2 \\
&\preceq 1_{\mathbb{A}} \| \sum_{k=m}^{n} |(d(x,Tx))^{1/2} a^k|^2 \| \\
&\preceq 1_{\mathbb{A}} \sum_{k=m}^{n} \|(d(x,Tx))^{1/2}\|^2 \|a^k\|^2 \\
&= 1_{\mathbb{A}} \|(d(x,Tx))^{1/2}\|^2 \sum_{k=m}^{n} \|a^2\|^k.
\end{aligned}$$

Since $\|a\| < 1$ and $m < n$, therefore as

$$m \longrightarrow \infty \text{ we must have } n \longrightarrow \infty.$$

So the fact that

$$\sum_{k=m}^{n} \|a^2\|^k$$

is an infinite geometric series completes the proof of (3.6). Further,

$$m \longrightarrow \infty \Rightarrow \|a\|^m \longrightarrow 0$$

and hence (3.7) follows from (3.6). □

Theorem 3.1.13.
Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space and $T : X \to X$ be a C^*-algebra valued contractive type mapping. That is, $\exists x \in X,\ a \in \mathbb{A}$ such that

$$d(Ty, T^2y) \preceq a^* d(y, Ty)a \ \text{ with } \|a\| < 1 \ \text{ for every } y \in \mathcal{O}_T(x).$$

Then

(A1) $\exists\ x_0 \in X$ such that the sequence $T^n x$ in $\mathcal{O}_T(x)$ converges to x_0,

(A2) $d(T^n x, x_0) \preceq \dfrac{\|a\|^{2n}}{1 - \|a\|} \|d(x, Tx)^{\frac{1}{2}}\|^2\, 1_{\mathbb{A}}$

(A3) x_0 is a fixed point of T if and only if the map

$$G(x) = d(x, Tx)$$

is $T-$orbitally lower semicontinuous at x_0 with respect to $\mathbb{A}$.

Proof.
If $\mathbb{A} = \{0_{\mathbb{A}}\}$ then there is nothing to prove. We, therefore, assume that $\mathbb{A}$ is a nontrivial C^*-algebra.

(A1):
Since the above contractive condition holds for each element of $\mathcal{O}_T(x)$ and $\|a\| < 1$, it follows that:

$$\begin{aligned} d(T^2x, T^3x) &= d(T(Tx), T^2(Tx)) \\ &\preceq a^* d(Tx, T(Tx))a \\ &= a^* d(Tx, T^2x)a \\ &\preceq a^* a^* d(x, Tx)aa \\ &= (a^*)^2 d(x, Tx)a^2. \end{aligned}$$

Continuing in this way, one can show that

$$d(T^n x, T^{n+1}x) \leq (a^*)^n d(x, Tx) a^n. \tag{3.8}$$

Let $\{T^n x\}$ be a sequence in $\mathcal{O}_T(x)$.

Then for $m < n$, we have from the triangle inequality and (3.8) that

$$\begin{aligned} & d(T^{n+1}x, T^m x) \\ & \quad \preceq \; d(T^m x, T^{m+1}x) + d(T^{m+1}x, T^{m+2}x) + \cdots + d(T^n x, T^{n+1}x) \\ & \quad \preceq \; (a^*)^m d(x, Tx) a^m + (a^*)^{m+1} d(x, Tx) a^{m+1} + \cdots + (a^*)^n d(x, Tx) a^n \\ & \quad = \; \sum_{k=m}^{n} (a^*)^k d(x, Tx) a^k \end{aligned}$$

Therefore, it follows from (3.7) of Lemma 3.1.12) that as $m \longrightarrow \infty$, we have

$$d(T^{n+1}x, T^m x) \longrightarrow 0_{\mathbb{A}}$$

This shows that $\{T^n x\}$ is a Cauchy sequence in $\mathcal{O}_T(x) \subset X$ with respect to $\mathbb{A}$. Since $(X, \mathbb{A}, d)$ is complete, there exists some $x_0 \in X$ such that

$$T^n x \longrightarrow x_0,$$

with respect to $\mathbb{A}$. This completes the proof of (A1).

(A2):

It follows again from the triangle inequality and (3.11) that

$$\begin{aligned}
&d(T^n x, T^{n+m}x)\\
&\quad\preceq\ d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + \cdots + d(T^{n+m-1}x, T^{n+m}x)\\
&\quad\preceq\ (a^*)^n d(x,Tx)a^n + (a^*)^{n+1} d(x,Tx)a^{n+1} + \cdots\\
&\qquad\ \cdots + (a^*)^{n+m-1} d(x,Tx)a^{n+m-1}\\
&\quad=\ \sum_{k=n}^{n+m-1} (a^*)^k d(x,Tx)a^k\\
&\quad\preceq\ \|(d(x,Tx))^{1/2}\|^2 \frac{\|a\|^{2n}}{1-\|a\|} 1_{\mathbb{A}}
\end{aligned}$$

The last inequality follows from (3.6) of Lemma 3.1.12).

Now as $m \longrightarrow \infty$, we get (A2).

(A3):

To prove (A3), if

$$Tx_0 = x_0$$

and x_n is a sequence in $\mathcal{O}_T(x)$ with $x_n \longrightarrow x_0$ with respect to $\mathbb{A}$,

then

$$\begin{aligned}
\|G(x_0)\| &= \|d(Tx_0, x_0)\|\\
&= 0\\
&\leq \liminf G(x_n).
\end{aligned}$$

Conversely, if G is T-orbitally lower-semi-continuous at x_0 then

$$\begin{aligned} \|G(x_0)\| &= \|d(x_0, Tx_0) \leq \liminf \|G(T^n x)\| \\ &= \liminf \|d(T^n x, T^{n+1} x)\| \\ &\leq \liminf \|a\|^{2n} \|d(x, Tx)\| \\ &= 0. \end{aligned}$$

This implies that

$$d(x_0, Tx_0) = 0_{\mathbb{A}}.$$

Thus T has a fixed point. □

Remark 3.1.14.

Note that:

1. By taking $\mathbb{A} = \mathbb{R}$, we see that the main result of [48] follows immediately from Theorem 3.1.19.
2. Theorem 2.5.22 is a special case of Theorem 3.1.19 except for the uniqueness of fixed point of the mapping involved.

Following examples show that our result properly generalizes Theorem 2.5.22.

Example 3.1.15. Let $X = [-1, 1]$ and $\mathbb{A} = \mathbb{R}$ with the usual metric. Let the map $T : X \to X$ be defined by

$$T(x) = \begin{cases} \frac{x}{4} & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}$$

Then for all $x \geq 0$ and $a = \frac{1}{2}$

$$d(Tx, T^2 x) \leq a^* d(x, Tx) a$$

and for all $x < 0$,

$$d(Tx, T^2 x) \leq a^* d(x, Tx) a,$$

where a is such that $\|a\| < 1$.

Define $f : X \to \mathbb{A}$ by

$$f(x) = d(x, Tx).$$

Now

$$\liminf_{x \to 0} f(x) = f(0) = 0,$$

so f is T-orbitally lower semi continuous at zero. Therefore all conditions of our result are satisfied and 0 is a fixed point of T.

Note that in this example the C^*-algebra $\mathbb{A}$ is the set of real numbers $\mathbb{R}$ and therefore the C^*-valued metric becomes the trivial real-valued metric on X. The main theorem of [62] is not applicable, since T is not continuous at zero.

For the non-trivial C^*-valued metric, we give the following simple example.

Example 3.1.16.
Take $X = [-1, 1]$ and $\mathbb{A} = \mathbb{R}^2$ with the C^*-valued metric d defined by

$$d(x, y) = (|x - y|, 0).$$

Define $T : X \to X$ by

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \geq 0 \\ 1 & \text{if } x < 0. \end{cases}$$

T is not continuous at 0. Then for $0 < x < 1$ we have

$$\begin{aligned} d(Tx, T^2x) &= d(\frac{x}{2}, \frac{x}{4}) \\ &= (|\frac{x}{4}|, 0) \\ &= (\frac{1}{\sqrt{2}}, 0)(\frac{x}{2}, 0)(\frac{1}{\sqrt{2}}, 0) \\ &= a^* d(x, Tx) a \end{aligned}$$

where $a = \left(\frac{1}{\sqrt{2}}, 0\right)$ and $\|a\| < 1$. Hence T is contractive type mapping on X.
If x_n is any sequence in $\mathcal{O}_T(x)$, then

$$x_n \to 0 \text{ with respect to } \mathbb{A},$$

Now the function $f : X \to \mathbb{A}$ defined by

$$f(x) = d(x, Tx) = (|\frac{x}{2}|, 0)$$

is T-orbitally lower semi continuous at zero. Therefore all conditions of our result are satisfied and 0 is a fixed point of T.

Example 3.1.17.
Consider the C^*-algebra $\mathbb{A} = \mathbb{R}^2$ with component-wise multiplication in $\mathbb{R}^2$ and let the C^*-valued metric $d : X \times X \to \mathbb{A}$ be as given in the above example and the ordering $\preceq$ be as given by (3.3) of Example 3.1.2. Let $X = [-1, 1]$ and define $T : X \to X$ by

$$T(x) = \begin{cases} \frac{x^2}{4} & \text{if } x \geq 0 \\ 1 & \text{if } x < 0. \end{cases}$$

Taking $x \in X$ such that $0 < x < 1$ we have

$$O_T(x) = \left\{ x, \frac{x^2}{4}, \frac{x^4}{4^3}, \ldots \right\}.$$

Further for any $y \in \mathcal{O}_T(x)$, it follows that

$$\begin{aligned} d(Ty, T^2y) &= d\left(\frac{y^2}{4}, \frac{y^4}{64}\right) \\ &= \frac{1}{4}\left(\frac{16y^2 - y^4}{16}, 0\right) \\ &\preceq \left(\frac{1}{2}, 0\right)\left(y - \frac{y^2}{4}, 0\right)\left(\frac{1}{2}, 0\right) \\ &= a^* d(y, Ty) a \end{aligned}$$

where $a = (\frac{1}{2}, 0)$ and hence T is contractive type mapping on X. Observe that any sequence $\{x_n\}$ in $\mathcal{O}_T(x)$ converges to 0.

Moreover, the function $G : X \to \mathbb{A}$ defined by

$$G(x) = d(x, Tx)$$

is T-orbitally lower semi continuous at 0. Therefore, all conditions of Theorem 3.1.19 are satisfied and 0 is the fixed point of T. Note that Theorem 2.5.22 is not applicable here, since T is not continuous at 0 with respect to $\mathbb{A}$.

Example 3.1.18.
Take $X = [-1, 1] \times [-1, 1]$ and $\mathbb{A} = \mathbb{R}^2$ with component-wise multiplication in $\mathbb{R}^2$. Define the metric $d : X \times X \to \mathbb{R}^2$ by

$$d(x, y) = (|x_1 - y_1|, |x_2 - y_2|) \ ,$$

for all $x, y \in X$ with $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Let the partial ordering $\preceq$ on the elements of $\mathbb{A}$ be as given in (3.3).

Define $T : X \to X$ by

$$Tx = T(x_1, x_2) = \begin{cases} (\frac{x_1}{2}, \frac{x_2}{2}) & \text{if } x_1,\, x_2 \geq 0 \\ (1, 0) & \text{otherwise.} \end{cases}$$

Clearly T is not continuous at $(0, 0) \in X$.

Taking $x = (x_1, x_2) \in X$ such that $0 < x_1, x_2 < 1$, we have

$$\mathcal{O}_T((x_1, x_2)) = \{(x_1, x_2), \left(\frac{x_1}{2}, \frac{x_2}{2}\right), \left(\frac{x_1}{4}, \frac{x_2}{4}\right), \ldots\}.$$

Choose an arbitrary element $y = (y_1, y_2) \in \mathcal{O}_T(x)$.

We have

$$\begin{aligned} d(Ty, T^2y) &= d\left(\left(\frac{y_1}{2}, \frac{y_2}{2}\right), \left(\frac{y_1}{4}, \frac{y_2}{4}\right)\right) \\ &= \left(\frac{y_1}{4}, \frac{y_2}{4}\right) \\ &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\left(\frac{y_1}{2}, \frac{y_2}{2}\right)\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\ &= a^* d(y, Ty) a \end{aligned}$$

where $a = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

It follows that T is contractive type mapping.

Now if x_n is any sequence in $\mathcal{O}_T(x)$, then

$$x_n \to (0,0),$$

and $f : X \to \mathbb{A}$ defined by $f(x) = d(x, Tx)$ is T-orbitally lower semi continuous at $(0,0)$, and $(0,0)$ is the fixed point of T.

Our next result is essentially extracted from the proof of Theorem 2.5.26 of [62].

Theorem 3.1.19. Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space and $T : X \to X$ be a mapping which satisfies for all $y \in \mathcal{O}_T(x)$

$$d(Ty, T^2y) \preceq ad(y, T^2y) \tag{3.9}$$

with $\|a\| \leq \dfrac{1}{2}$ and $a \in \mathbb{A}'_+$, where

$$\mathbb{A}'_+ = \{a \in \mathbb{A}_+ \mid ab = ba \text{ for all } b \in \mathbb{A}_+\}.$$

Then

(A1) $\exists\, x_0 \in X$ such that the sequence $T^n x$ converges to x_0,

(A2) $d(T^n x, x_0) \preceq \dfrac{\|h\|^n}{1 - \|h\|} \|d(x, Tx)\|\, 1_{\mathbb{A}}$ where $a(1_{\mathbb{A}} - a^{-1}) = h$.

(A3) x_0 is a fixed point of T if and only if $G(x) = d(x, Tx)$ is T-orbitally lower semi continuous at x_0 with respect to $\mathbb{A}$.

Proof.
If $\mathbb{A} = \{0_{\mathbb{A}}\}$ then there is nothing to prove. Assume that $\mathbb{A} \neq \{0_{\mathbb{A}}\}$.

(A1)

Let $x \in X$ and consider the orbit $\mathcal{O}_T(x)$. Since Condition (3.9) holds for each element of $\mathcal{O}_T(x)$ and $\|a\| < 1$, it follows that:

$$\begin{aligned}
d(T^{n+1}x, T^n x) &= d(T^{n+1}x, T(T^{n-1}x)) \\
&\preceq ad(T^{n+1}x, T^{n-1}x) \\
&\preceq a[d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] \\
&= ad(T^{n+1}x, T^n x) + ad(T^n x, T^{n-1}x).
\end{aligned}$$

Thus

$$d(T^{n+1}x, T^n x) - ad(T^{n+1}x, T^n x) \preceq ad(T^n x, T^{n-1}x).$$

Which implies that

$$(1_{\mathbb{A}} - a)d(T^{n+1}x, T^n x) \preceq ad(T^n x, T^{n-1}x)$$

Since $a \in \mathbb{A}'_+$ with $\|a\| < \dfrac{1}{2}$. By Lemma 2.5.25 we have

$$(1_{\mathbb{A}} - a)^{-1} \in \mathbb{A}'_+$$

and also

$$a(1_{\mathbb{A}} - a)^{-1} \in \mathbb{A}'_+ \text{ with } \|a(1_{\mathbb{A}} - a)^{-1}\| < 1.$$

Therefore,

$$d(T^{n+1}x, T^n x) \preceq a(1_{\mathbb{A}} - a)^{-1}d(T^n x, T^{n-1}x). \tag{3.10}$$

Rewriting the above inequality with $a(1_{\mathbb{A}} - a)^{-1} = h$ as

$$d(T^{n+1}x, T^n x) \preceq hd(T^n x, T^{n-1}x). \tag{3.11}$$

Writing $d(x, Tx) = p$ and let $\{T^n x\}$ be a sequence in $\mathcal{O}_T(x)$. Then from the triangle inequality and (3.11), for $m < n$ we have

$$
\begin{aligned}
d(T^{n+1}x, T^m x) & \\
& \preceq d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x) + \cdots + d(T^{m+1}x, T^m x) \\
& \preceq (h^n + h^{n-1} + \cdots + h^m) d(Tx, x) \\
& = \sum_{k=m}^{n} h^k d(x, Tx) \\
& = \sum_{k=m}^{n} h^{k/2} h^{k/2} p^{1/2} p^{1/2} \\
& = \sum_{k=m}^{n} (h^{k/2} p^{1/2})^* (h^{k/2} p^{1/2}) \\
& = \sum_{k=m}^{n} |h^{k/2} p^{1/2}|^2 \\
& \preceq \left\| \sum_{k=m}^{n} |h^{k/2} p^{1/2}|^2 \right\| 1_{\mathbb{A}}
\end{aligned}
$$

which yields to

$$
\begin{aligned}
d(T^{n+1}x, T^m x) \quad & \preceq \quad \sum_{k=m}^{n} \|h^{k/2}\|^2 \|p^{1/2}\|^2 1_{\mathbb{A}} \\
& \preceq \quad \|p^{1/2}\|^2 \sum_{k=m}^{n} \|h^{k/2}\|^2 1_{\mathbb{A}} \\
& \preceq \quad \|p^{1/2}\|^2 \frac{\|h\|^m}{1 - \|h\|} 1_{\mathbb{A}} \\
& \longrightarrow \quad 0_{\mathbb{A}} \text{ as } m \longrightarrow \infty.
\end{aligned}
$$

This shows that $\{T^n x\}$ is a Cauchy sequence in $\mathcal{O}_T(x) \subset X$ with respect to $\mathbb{A}$. Since $(X, \mathbb{A}, d)$ is a complete C^*-algebra valued metric space, there exists $x_0 \in X$ such that $T^n x \longrightarrow x_0$. This completes the proof of (A1).

To prove (A2) we proceed as follows:

(A2):

It follows again from the triangle inequality and Condition (3.11) that:

$$\begin{aligned}
d(T^n x, T^{n+m}x) &\preceq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) \\
&\quad + \cdots + d(T^{n+m-1}x, T^{n+m}x) \\
&\preceq (h)^n d(x, Tx) + (h)^{n+1} d(x, Tx) \\
&\quad + \cdots + (h)^{n+m-1} d(x, Tx) \\
&= \sum_{k=n}^{n+m-1} (h)^k d(x, Tx) \\
&\preceq \frac{\|h\|^n}{1 - \|h\|} d(x, Tx) 1_{\mathbb{A}}
\end{aligned}$$

letting $m \longrightarrow \infty$, we get from (A1) that

$$T^{n+m}x \longrightarrow x_0,$$

and hence

$$d(T^n x, x_0) \preceq \frac{\|h\|^n}{1 - \|h\|} \|d(x, Tx)\| \, 1_{\mathbb{A}}$$

This proves (A2).

(A3):

To prove (A3), if $Tx_0 = x_0$ and $\{x_n\}$ is a sequence in $\mathcal{O}_T(x)$ such that

$$x_n \longrightarrow x_0 \text{ with respect to } \mathbb{A},$$

then

$$\begin{aligned}
|G(x_0)\| &= \|d(Tx_0, x_0)\| \\
&= 0 \\
&\leq \liminf \|G(x_n)\|.
\end{aligned}$$

Conversely, if G is T-orbitally lower semi continuous at x_0 then

$$\begin{aligned}\|G(x_0)\| &= \|d(x_0, Tx_0)\| \leq \liminf \|G(T^n x)\| \\ &= \liminf \|d(T^n x, T^{n+1} x)\| \\ &\leq \liminf \|h\|^n \|d(x, Tx)\| \\ &= 0_{\mathbb{A}}.\end{aligned}$$

This implies that

$$d(x_0, Tx_0) = 0_{\mathbb{A}},$$

i.e., $x_0 = Tx_0$.

Hence T has a fixed point. □

3.2 Application

In this section we provide the existence result for an integral equation as an application of C^*-valued contractive type mappings on complete C^*-valued metric spaces.

Let E be a Lebesgue measurable set, $X = L^\infty(E)$, and $H = L^2(E)$.

We denote the set of all bounded linear operators on Hilbert space H by $L(H)$.

With the usual operator norm, $L(H)$ is a C^*-algebra.

For $S, T \in X$, define $d : X \times X \to L(H)$ by

$$d(T, S) = \pi_{|T-S|},$$

where

$$\pi_h : H \to H$$

is the multiplication operator given by

$$\pi_h(\phi) = h \cdot \phi$$

for $\phi \in H$.

Then $(X, L(H), d)$ is a complete C^*-valued metric space [62].

Example 3.2.1. Let E, X, H, and the metric d be as above. Suppose that

1. $K : E \times E \times \mathbb{R} \to \mathbb{R}$, and

   let T be a self mapping on X,

2. there exists a continuous function $\phi : E \times E \to \mathbb{R}$ and $\alpha \in (0, 1)$ such that for every $x \in X$, $y \in \mathcal{O}_T(x)$, and $t, s \in E$,we have

$$|K(t, s, x(s)) - K(t, s, y(s))| \leq \alpha|\phi(t, s)(x(s) - y(s))|. \tag{3.12}$$

3. $\sup_{t \in E} \int_E |\phi(t, s)|ds \leq 1$.

Then the integral equation

$$x(t) = \int_E K(t, s, x(s))ds, \quad t \in E$$

has a solution.

Proof.

Here $(X, L(H), d)$ is a complete C^*-valued metric space with respect to $L(H)$.

Let $T : X \to X$ be

$$Tx(t) = \int_E K(t, s, x(s))ds, \quad t \in E$$

Let $Tx = y$, then

$$\begin{aligned}
\|d(Tx, T^2x)\| &= \|d(Tx, Ty)\| \\
&= \|\pi_{|Tx-Ty|}\| \\
&= \sup_{\|h\|=1} \langle \pi_{|Tx-Ty|}h, h\rangle, \qquad \text{for any } h \in H \\
&= \sup_{\|h\|=1} \int_E \left[\left| \int_E (K(t,s,x(s)) - K(t,s,y(s)))ds \right| \right] h(t)\overline{h(t)}dt \\
&\leq \sup_{\|h\|=1} \int_E \left[\left| \int_E (K(t,s,x(s)) - K(t,s,y(s)))ds \right| \right] |h(t)|^2 dt \\
&\leq \sup_{\|h\|=1} \int_E \left[\int_E |k\phi(t,s)(x(s) - y(s))| ds \right] |h(t)|^2 dt \\
&\leq k \sup_{\|h\|=1} \int_E \left[\int_E |\phi(t,s)| ds \right] |h(t)|^2 dt \cdot \|x-y\|_\infty \\
&\leq k \sup_{t\in E} \int_E |\phi(t,s)| ds \cdot \sup_{\|h\|=1} \int_E |h(t)|^2 dt \cdot \|x-y\|_\infty \\
&\leq k\|x-y\|_\infty \\
&= \|a\|\|d(x,y)\| = \|a\|\|d(x,Tx)\|
\end{aligned}$$

setting $a = kI$, we have $a \in L(H)_+$ and $\|a\| = k$. Thus all the conditions of Theorem 3.1.19 holds and hence the conclusion. □

3.3 Caristi type C^*-valued Contractions

In this section we present the extension of Caristi's fixed point theorem for mappings defined on C^*-algebra valued metric spaces. We prove the existence of fixed point using the concept of minimal element in C^*-algebra valued metric space by introducing the notion of partial order on X. Taking advantage offered by this framework, we extend the Caristi's fixed point theorem in context of C^*-algebra valued metric space.

We begin this section by introducing the notion of lower semi continuity in the context of C^*-algebra valued metric spaces.

Definition 3.3.1.
Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space. A mapping $\phi\colon X \to \mathbb{A}$ is said to be lower semi continuous at x_0 with respect to $\mathbb{A}$ if

$$\|\phi(x_0)\| \leq \lim_{x \to x_0} \inf \|\phi(x)\|$$

Example 3.3.2.
Let $X = [-1, 1]$ and $\mathbb{A} = \mathbb{R}^2$ be the C^*-algebra with $\|(a_1, a_2)\| = \sqrt{|a_1|^2 + |a_2|^2}$. Define an order $\preceq$ on $\mathbb{A}$ as follows:

$$(x_1, y_1) \preceq (x_2, y_2) \;\Leftrightarrow\; x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

where "$\leq$" is the usual order on the elements of $\mathbb{R}$. It is easy to see that $\preceq$ is a partial order on $\mathbb{A}_+$. Consider $d\colon X \times X \to \mathbb{A}$ defined by

$$d(x, y) = (|x - y|, 0),$$

then clearly $(X, \mathbb{A}, d)$ is a C^*-algebra valued metric space. Define a map

$$\phi\colon X \to \mathbb{A}, \quad \phi(x) = \begin{cases} (\frac{x}{2}, 0) & \text{if } x \geq 0 \\ (1, 0) & \text{otherwise.} \end{cases}$$

Then it is easy to see that ϕ is lower semi continuous at $x_0 = 0$.

It is straightforward to prove the following lemma:

Lemma 3.3.3.
Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space and let $\phi\colon X \to \mathbb{A}_+$ be a map. Define the order $\preceq_\phi$ on X by

$$x \preceq_\phi y \Longleftrightarrow d(x, y) \preceq \phi(y) - \phi(x) \text{ for any } x, y \in X. \tag{3.13}$$

Then $\preceq_\phi$ is a partial order on X.

Theorem 3.3.4.
Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space and $\phi\colon X \to \mathbb{A}_+$ be a lower semi-continuous map. Then $(X, \preceq_\phi)$ has a minimal element, where $\preceq_\phi$ is defined by (3.13).

As a consequence of the above theorem we have the following fixed-point result.

Theorem 3.3.5.
Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space and $\phi\colon X \to \mathbb{A}_+$ be a lower semi continuous map. Let $T\colon X \to X$ be such that for all $x \in X$

$$d(x, Tx) \preceq \phi(x) - \phi(Tx). \tag{3.14}$$

Then T has at least one fixed point.

Example 3.3.6.
Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{R}^2$ be a C^*-algebra with the partial order as given in Example 3.3.2. Define $d\colon X \times X \to \mathbb{A}$ by

$$d(x, y) = (|x - y|, 0).$$

Let the continuous map $\phi\colon X \to \mathbb{A}_+$ be

$$\phi(x) = (x, 0),$$

and $T\colon X \to X$ be given by the formula

$$T(x) = x^2.$$

Then it is easy to see that all the conditions of Theorem 3.3.5 are satisfied and T has a fixed point.

Note that contractive theorem stated in [62] is not applicable here, since contractive condition (2.6) does not hold.

Chapter 4

C^*-valued b-Metric Spaces

In this chapter we introduce the notion of C^*-valued b-metric space to generalize the notion of C^*-valued metric space. We then extend the work of Ma. et. al. [62] in this new setting. Our results not only generalize the fixed point theorem by Ma. et. al. [62] but also the fixed point theorems by Czerwick [27]. We have also applied our result to establish a solution of an integral equation in this new setting.

4.1 C^*-valued b-Metric Spaces

Motivated by Definition 2.3.1, we extend the notion of b-metric space in the setting of C^*-algebras as follows.

Definition 4.1.1. (**C^*-valued b-Metric Space**)
Let X be a non empty set and $\mathbb{A}$ be a unital C^*-algebra. Let $b \in \mathbb{A}$ be such that $\|b\| \geq 1$. A C^*-valued b-metric on X is a mapping $d_b : X \times X \to \mathbb{A}$ satisfying the following conditions:

BM1: $d_b(x, y) \succeq 0_{\mathbb{A}}$ for all $x, y \in X$ and $d_b(x, y) = 0_{\mathbb{A}} \Leftrightarrow x = y$;

BM2: $d_b(x, y) = d_b(y, x)\ \forall\ x, y \in X$ (symmetry);

BM3: $d_b(x, z) \preceq b\,[d_b(x, y) + d_b(y, z)]\ \forall\ x, y, z \in X.$ (triangle inequality).

The triplet $(X, \mathbb{A}, d_b)$ is called a C^*-valued b-metric space with the coefficient b.

Remark 4.1.2.
Note that:

1. If we take $\mathbb{A} = \mathbb{R}$, the new notion of C^*-valued b-metric space becomes equivalent to Definition 2.3.1 of the real b-metric space.

2. If we take $b = 1_{\mathbb{A}}$ in Definition 4.1.1, then d_b becomes the usual C^*-valued-metric as defined in [62].

Thus the class of C^*-valued b-metric spaces is effectively larger than that of the ordinary C^*-valued metric spaces. That is, every C^*-valued metric space is a b-metric space,but the converse need not be true. This is illustrated in the following example.

Example 4.1.3.
Let $X = \ell_p$, the set of sequences $\{x_n\}$ in $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty \text{ and } 0 < p < 1.$$

Let $\mathbb{A} = M_2(\mathbb{R})$. For $x = x_n, y = y_n \in \ell_p$, define $d_b : X \times X \to \mathbb{A}$ as

$$d_b(x, y) = \begin{pmatrix} (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}} & 0 \\ 0 & (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}} \end{pmatrix}.$$

Then it is easy to see that d is C^*-valued b-metric space with coefficient

$$b = \begin{pmatrix} 2^{\frac{1}{p}} & 0 \\ 0 & 2^{\frac{1}{p}} \end{pmatrix} \text{ with } \|b\| = \sqrt{2}\, 2^{\frac{1}{p}}.$$

The claim follows from the following observation in [28]

$$\left(\sum_{n=1}^{\infty} |x_n - z_n|^p\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p}} \left[\left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n - z_n|^p\right)^{\frac{1}{p}}\right].$$

Note that, here $(X, \mathbb{A}, d_b)$ is not a usual C^*-valued metric space.

Given a C^*-valued b-metric space $(X, \mathbb{A}, d_b)$, then the following are natural deductions from the corresponding notions in C^*-valued metric spaces.

1. A sequence $\{x_n\}$ in X is said to be convergent, with respect to $\mathbb{A}$, to a point $x \in X$, if and only if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|d_b(x_n, x)\| < \epsilon \text{ for all } n > N.$$

   We write it as $\lim_{n\to\infty} x_n = x$.

2. If for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|d_b(x_n, x_m)\| < \epsilon$ for all $n, m > N$, then the sequence $\{x_n\}$ is called a Cauchy sequence with respect to $\mathbb{A}$.

3. The triplet $(X, \mathbb{A}, d_b)$ is said to be a complete C^*-valued b-metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

4.2 Fixed Point Theorems for C^*-valued b- Metric Spaces

We begin this section by extending the C^*-contraction condition in the context of C^*-valued b-metric space.

Definition 4.2.1. (**Contraction**)
Let $(X, \mathbb{A}, d_b)$ be a C^*-valued b-metric space. A mapping $T : X \to X$ is said to be a contraction if there exists an $a \in \mathbb{A}$ such that

$$d_b(Tx, Ty) \preceq a^* d_b(x, y)a \quad \text{with } \|a\| < 1 \quad \text{for every } x, y \in X. \tag{4.1}$$

Example 4.2.2.
Let $X = [0, \infty)$, $\mathbb{A} = \mathbb{R}^2$ with partial order $\preceq$ on $\mathbb{A}$ given by

$$(x_1, y_1) \preceq (x_2, y_2) \Leftrightarrow x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

Define $d_b : X \times X \to \mathbb{A}$ as

$$d_b(x, y) = ((x - y)^2, 0),$$

then d_b is C^*-valued b-metric with coefficient $(2, 0)$ and $(X, \mathbb{A}, d_b)$ is a C^*-valued b-metric space. Define $T : X \to X$ by

$$Tx = \frac{x}{3} + 5\,,$$

then it is easy to see that T is a contraction with $a = (\frac{1}{3}, 0)$. In fact, we have

$$\begin{aligned} d_b(Tx, Ty) &= \big((Tx - Ty)^2, 0\big) \\ &= \left(\left(\frac{x}{3} - \frac{y}{3}\right)^2, 0\right) \\ &= \left(\frac{1}{3}, 0\right) d_b(x, y) \left(\frac{1}{3}, 0\right). \end{aligned}$$

we are now in position to extend Theorem (Ma et al) in case of C^*-valued b-metric space.

Theorem 4.2.3.
Let $(X, \mathbb{A}, d_b)$ be a complete C^*-valued b-metric space with coefficient b. Let $T : X \to X$ be a contraction with the contraction constant a, such that $\|b\|\|a\|^2 < 1$. Then T has a unique fixed point in X.

Proof.
If $\mathbb{A} = \{0_{\mathbb{A}}\}$ then there is nothing to prove. Assume that $\mathbb{A} \neq \{0_{\mathbb{A}}\}$.

Choose $x_0 \in X$ and define the sequence $\{x_n\}$ inductively by the iterative scheme as

$$x_{n+1} = Tx_n.$$

Then it follows that

$$x_n = T^n x_0$$

for $n = 0, 1, 2, \ldots$.

Since T is a contraction, it follows from (4.1) that

$$\begin{aligned}
d_b(x_n, x_{n+1}) &= d_b(Tx_{n-1}, Tx_n) \\
&\preceq a^* d_b(x_{n-1}, x_n) a \\
&= a^* d_b(Tx_{n-2}, Tx_{n-1}) a \\
&\preceq (a^*)^2 d_b(x_{n-2}, x_{n-1}) a^2 \\
&\preceq (a^*)^3 d_b(x_{n-3}, x_{n-2}) a^3 \preceq (a^*)^n d_b(x_0, x_1) a^n = (a^*)^n D a^n
\end{aligned}$$

where $D = d_b(x_0, x_1)$.

Now suppose that $m > n$, then by the triangle inequality for b-metric, we have

$$\begin{aligned}
d_b(x_n, x_m) &\preceq b\, d_b(x_n, x_{n+1}) + b^2 d_b(x_{n+1}, x_{n+2}) + \cdots + b^{m-n-1} d_b(x_{m-2}, x_{m-1}) \\
&+ b^{m-n-1} d_b(x_{m-1}, x_m) \\
&\preceq b(a^*)^n D a^n + b^2 (a^*)^{n+1} D a^{n+1} + \cdots + b^{m-n-1} (a^*)^{m-2} D a^{m-2} \\
&+ b^{m-n-1} (a^*)^{m-1} D a^{m-1} \\
&= b[(a^*)^n D a^n + b(a^*)^{n+1} D a^{n+1} + \cdots + b^{m-n-2} (a^*)^{m-2} D a^{m-2}] \\
&+ b^{m-n-1} (a^*)^{m-1} D a^{m-1}.
\end{aligned}$$

Using the summation notation on right hand side, we get

$$\begin{aligned}
d_b(x_n, x_m) &= b\sum_{k=n}^{m-2} b^{k-n}(a^*)^k D a^k + b^{m-n-1}(a^*)^{m-1} D a^{m-1} \\
&= b\sum_{k=n}^{m-1} b^{k-n}(a^*)^k D^{\frac{1}{2}} D^{\frac{1}{2}} a^k + b^{m-n-1}(a^*)^{m-1} D^{\frac{1}{2}} D^{\frac{1}{2}} a^{m-1} \\
&= b\sum_{k=n}^{m-2} b^{k-n}(D^{\frac{1}{2}} a^k)^*(D^{\frac{1}{2}} a^k) + b^{m-n-1}(D^{\frac{1}{2}} a^{m-1})^*(D^{\frac{1}{2}} a^{m-1}) \\
&= b\sum_{k=n}^{m-2} b^{k-n}|D^{\frac{1}{2}} a^k|^2 + b^{m-n-1}|D^{\frac{1}{2}} a^{m-1}|^2 \\
&\preceq \|b\sum_{k=n}^{m-2} b^{k-n}|D^{\frac{1}{2}} a^k|^2\| 1_{\mathbb{A}} + \|b^{m-n-1}|D^{\frac{1}{2}} a^{m-1}|^2\| 1_{\mathbb{A}} \\
&\preceq \|b\| \sum_{k=n}^{m-2} \|b^{k-n}\| \|D^{\frac{1}{2}}\|^2 \|a^k\|^2 1_{\mathbb{A}} + \|b^{m-n-1}\| \|D^{\frac{1}{2}}\|^2 \|a^{m-1}\|^2 1_{\mathbb{A}} \\
&\preceq \|b\| \sum_{k=n}^{m-2} \|b\|^{k-n} \|D^{\frac{1}{2}}\|^2 \|a^k\|^2 1_{\mathbb{A}} + \|b\|^{m-n-1} \|D^{\frac{1}{2}}\|^2 \|a^{m-1}\|^2 1_{\mathbb{A}} \\
&\preceq \|b\|^{1-n} \|D^{\frac{1}{2}}\|^2 \sum_{k=n}^{m-2} \|b\|^k \|a^2\|^k 1_{\mathbb{A}} + \|b\|^{-n} \|b\|^{m-1} \|D^{\frac{1}{2}}\|^2 \|a^{m-1}\|^2 1_{\mathbb{A}} \\
&\preceq \|b\|^{1-n} \|D^{\frac{1}{2}}\|^2 \sum_{k=n}^{m-2} (\|b\| \|a^2\|)^k 1_{\mathbb{A}} + \|b\|^{-n} \|D^{\frac{1}{2}}\|^2 (\|b\| \|a^2\|)^{m-1} 1_{\mathbb{A}} \\
&\longrightarrow 0_{\mathbb{A}} \quad \text{as } m,\, n \to \infty,
\end{aligned}$$

which follows from the observation that the summation in the first term is a geometric series and $\|b\|\|a^2\| < 1$ implies that both

$$(\|b\|\|a^2\|)^{m-1} \longrightarrow 0 \quad \text{and} \quad (\|b\|\|a^2\|)^{n-1} \longrightarrow 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in X with respect to $\mathbb{A}$. As $(X, \mathbb{A}, d)$ is complete, it follows that $x_n \to x \in X$, *i.e.*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = x.$$

We claim that x is a fixed point of T. In fact, from (BM3) and the contraction condition (4.1) we have:

$$\begin{aligned} 0_{\mathbb{A}} &\preceq d_b(Tx, x) \\ &\preceq b[d_b(Tx, Tx_n) + d_b(Tx_n, x)] \\ &\preceq b\, a^* d_b(x, x_n) a + d_b(x_{n+1}, x) \longrightarrow 0_{\mathbb{A}} \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence

$$Tx = x.$$

To prove the uniqueness of the fixed point x, suppose that y is another fixed point of T. Then again from the contraction condition (4.1), we have

$$0_{\mathbb{A}} \preceq d_b(x, y) = d_b(Tx, Ty) \preceq a^* d_b(x, y) a$$

Using the norm of $\mathbb{A}$, we have

$$\begin{aligned} 0 &\leq \|d_b(x, y)\| \\ &\leq \|a^* d_b(x, y) a\| \\ &\leq \|a^*\| \|d_b(x, y)\| \|a\| \\ &= \|a\|^2 \|d_b(x, y)\| \end{aligned}$$

which is possible only when

$$d_b(x, y) = 0_{\mathbb{A}}.$$

Hence

$$x = y.$$

□

Example 4.2.4.

The mapping T of Example 4.2.2 satisfies hypothesis of Theorem 4.2.3 and T has the unique fixed point $x = \dfrac{15}{2}$ in X.

Remark 4.2.5. Theorem 4.2.3 generalizes the following results.

1. By taking $\mathbb{A} = \mathbb{R}$, the C^*-valued b-metric become the b-metric and the Banach contraction principle in b-metric spaces follows immediately from Theorem 4.2.3.

2. Taking $b = 1$, [62, Theorem 2.1] becomes a special case of Theorem 4.2.3.

4.3 Application

As an application of fixed point of contractions on a complete C^*-valued b-metric spaces we provide the existence result for an integral equation.

Example 4.3.1.
Let E be a Lebesgue measurable set, $X = L^\infty(E)$, and $H = L^2(E)$. We denote by $L(H)$ the set of all bounded linear operators on defined on the Hilbert space H. With the usual operator norm, $L(H)$ is a C^*-algebra. For $S, T \in X$, define $d_b : X \times X \to L(H)$ by

$$d_b(T, S) = \mu_{(T-S)^2},$$

where $\mu_h : H \to H$ is the multiplication operator given by

$$\mu_h(\phi) = h \cdot \phi \text{ for } \phi \in H.$$

Working in the same lines as in [62, Example 2.1], one can show that $(X, L(H), d_b)$ is a complete C^*-valued b-metric space.

Suppose that for the map $K : E\times E\times\mathbb{R} \to \mathbb{R}$, there is some function $\phi : E\times E \to \mathbb{R}$ which is continuous and a real numer k in $(0, 1)$ such that for every $x,\ y \in X$ and $t, s \in E$,we have

$$|K(t, s, x(s)) - K(t, s, y(s))| \le k|\phi(t, s)(x(s) - y(s))|. \tag{4.2}$$

and

$$\sup_{t\in E} \int_E |\phi(t, s)|ds \le 1 \tag{4.3}$$

Then the integral equation give below

$$x(t) = \int_E K(t, s, x(s))ds, \quad t \in E \tag{4.4}$$

has a unique solution.

Proof.
Here $(X, L(H), d_b)$ is a complete C^*-valued b-metric space with respect to $L(H)$.

Let $T : X \to X$ be

$$Tx(t) = \int_E K(t, s, x(s))ds, \quad t \in E$$

Then

$$\begin{aligned}
\|d(Tx, Ty)\| &= \|\mu_{(Tx-Ty)^2}\| \\
&= \sup_{\|h\|=1} \langle \mu_{(Tx-Ty)^2} h, h \rangle, \qquad \text{for any } h \in H \\
&= \sup_{\|h\|=1} \int_E (Tx - Ty)^2 h(t)\overline{h(t)}ds \\
&= \sup_{\|h\|=1} \int_E \left[\int_E (K(t, s, x(s)) - K(t, s, y(s)))ds \right]^2 h(t)\overline{h(t)}dt \\
&\leq \sup_{\|h\|=1} \int_E \left[\int_E (K(t, s, x(s)) - K(t, s, y(s)))ds \right]^2 |h(t)|^2 dt \\
&\leq \sup_{\|h\|=1} \int_E k^2 \left[\int_E (\phi(t, s)(x(s) - y(s)))ds \right]^2 |h(t)|^2 dt \\
&\leq k^2 \sup_{\|h\|=1} \int_E \left[\int_E |\phi(t, s)|ds \right]^2 |h(t)|^2 dt \cdot \|(x - y)^2\|_\infty \\
&\leq k^2 \sup_{t \in E} \int_E |\phi(t, s)|^2 ds \cdot \sup_{\|h\|=1} \int_E |h(t)|^2 dt \cdot \|(x - y)^2\|_\infty \\
&\leq k^2 \|(x - y)^2\|_\infty \\
&= \|a\| \|d(x, y)\|
\end{aligned}$$

setting $a = kI$, we have $a \in L(H)_+$ and $\|a\| = k^2 < 1$. Thus all the conditions of Theorem 4.2.3 holds and hence the conclusion. $\square$

Chapter 5

Fixed Point Theorems for C^*-multivalued Contractions

Lot of work has been done on fixed points of multivalued functions. In this chapter we introduced the notions of bounded sets, closed sets with respect to a C^*-valued-metric space. We also introduced C^*-valued Hausdorff metric and the contraction condition of C^*-multivalued mappings. We proved that a C^*-multivalued contraction mapping on a complete C^*-valued-metric space has a fixed point, where $\mathbb{A}_+$ is totally ordered set which is the generalization for the result proved by Nadler [71] in the setting of C^*-algebra. In [71] he has proved some interesting results related to the fixed points for multivalued contraction mappings. Some of his results are given in [70]. Afterward a facinating and rich theory for fixed point of multivalued maps was developed which has application in control theory, convex optimazition and economics see [45].

Through out this chapter we will assume that $(X, \mathbb{A}, d)$ is a C^*-valued-metric space where X is a nonempty set, $\mathbb{A}$ is a C^*-algebra and d is a C^*-valued-metric on X.

5.1 C^*-Multivalued Contractions

Recall that if X is a non-empty set and T is a mapping from X to some collection of subsets of X then a point $x \in X$ is called a fixed point of T if $x \in Tx$.

In order to define C^*-valued Hausdorff metric we need the concept of closed and bounded sets in $(X, \mathbb{A}, d)$.

Definition 5.1.1. (**Neighbourhood**)
Let $(X, \mathbb{A}, d)$ be a C^*-valued-metric space, $A \subseteq X$ and let $r \succeq 0_{\mathbb{A}}$. A neighborhood of a point $x \in X$ with respect to $\mathbb{A}$ is defined as the set

$$\{y \in X : d(x, y) \preceq r\}.$$

Definition 5.1.2. (**Limit Point**)
A point $x \in X$ is said to be a limit point of A if every neighborhood of x contains at least one point of A other than x.

Definition 5.1.3. (**Closed Set**)
A subset A of a C^*-valued-metric space is said to be closed with respect to $\mathbb{A}$ if it contains all of its limit points.

Definition 5.1.4. (**Bounded Sets**)
Let $(X, \mathbb{A}, d)$ be a C^*-valued-metric space. A subset A of X is said to be bounded with respect to $\mathbb{A}$. If there exists $M \in \mathbb{A}_+$ and a point $y \in A$ such that for all $x, y \in A$,

$$d(x, y) \preceq M.$$

Remark 5.1.5.
When $\mathbb{A} = \mathbb{R}$, the C^*-valued-metric d becomes the standard real-valued-metric on X and the above definitions coincide with the standard definitions of neighborhood, closed sets and bounded sets in a metric space.

Definition 5.1.6. (**Distance of a Point from a Set**)
Suppose $(X, \mathbb{A}, d)$ be a C^*-valued-metric space and A is a non-empty subsets of X and $x \in X$. Further, assume that the range of the metric d is a totally ordered subset of $\mathbb{A}_+$. We define the C^*-distance from A to x with respect to $\mathbb{A}$ as follows

$$\operatorname{dist}(x, A) = \inf\{d(x, a) : a \in A\}. \tag{5.1}$$

The existence of inf in the above example is guaranteed from the assumption that d maps to a totally ordered subset of $\mathbb{A}_+$. Clearly, this distance is heavily dependent on the C^*-valued-metric d.

Example 5.1.7. Let $X = [-1, 1]$, $\mathbb{A} = \mathbb{R}^2$ with the component-wise operations of addition and multiplication and the ordering $\preceq$ be as given by (3.3) of Example 3.1.2. Define $d_1 : X \times X \to \mathbb{A}$ by

$$d_1(x, y) = (|x - y|, 0).$$

Then it is easy to see that $(X, \mathbb{A}, d_1)$ is a C^*-valued-metric space.

Let $A = [\frac{1}{4}, \frac{1}{2}] \subset X$. Then illustrate the above definition of C^*-valued distance of a point from a set, note that from (5.1) of Definition 5.1.6 we have

$$\begin{aligned} \mathrm{dist}(0.1, A) &= \inf\{d_1(0.1, a) : a \in [\frac{1}{4}, \frac{1}{2}]\} \\ &= \inf\{(|0.1 - a|, 0) \ : \ a \in [\frac{1}{4}, \frac{1}{2}]\} \\ &= (0.15, 0) \in \mathbb{R}^2. \end{aligned}$$

Whereas if we define $d_2 : X \times X \to \mathbb{R}^2$ by

$$d_2(x, y) = (|x - y|, |x - y|)$$

then again $(X, \mathbb{A}, d_2)$ is a C^*-valued-metric space and from (5.1) we now have

$$\begin{aligned} \mathrm{dist}(0.1, A) &= \inf\{d_2(0.1, a) : a \in [\frac{1}{4}, \frac{1}{2}]\} \\ &= \inf\{(|0.1 - a|, |0.1 - a|) \ : \ a \in [\frac{1}{4}, \frac{1}{2}]\} \\ &= (0.15, 0.15) \in \mathbb{R}^2. \end{aligned}$$

Corresponding to usual notion of the distance between the sets of a metric space we introduce the notion of C^*-valued distance between the sets of a C^*-valued-metric space $(X, \mathbb{A}, d)$ as follows.

Definition 5.1.8. (**Hausdorff Distance**)
Let $(X, \mathbb{A}, d)$ be a C^*-valued-metric space and assume that the range of d is a totally ordered subset of $\mathbb{A}_+$. Let $CB(X)$ be the collection of all closed and bounded subsets of X. For each $A, B \in CB(X)$, define $\mathcal{H} : CB(X) \times CB(X) \to \mathbb{A}$

$$\mathcal{H}(A, B) = \max\{\sup\{\mathrm{dist}(b, A) : b \in B\}, \{\sup \mathrm{dist}(a, B) : a \in A\}\}. \tag{5.2}$$

The distance $\mathcal{H}(A,B)$ is called the C^*-valued Hausdorff distance between the sets in $CB(X)$ generated by the C^*-valued-metric d.

Remark 5.1.9. If we set $\mathbb{A}=\mathbb{R}$ in Definition 5.1.8 then $\mathcal{H}(A,B)$ coincides with the standard real-valued Hausdorff distance between the subsets A and B of X.

In order to prove the next theorem we need the following lemmas.

Lemma 5.1.10.
Let $(X,\mathbb{A},d)$ be a C^*-valued-metric space and assume that the range of d is a totally ordered subset of $\mathbb{A}_+$. Let $A,\ B\in CB(X)$ and let $a\in A$. If $x\succeq 0_{\mathbb{A}}$, then there exists $b\in B$ such that

$$d(a,b)\preceq \mathcal{H}(A,B)+x.$$

Proof.
Let $a\in A$ and $x\succeq 0_{\mathbb{A}}$ then there exists $b\in B$ such that

$$\begin{aligned} d(a,b) &\preceq \operatorname{dist}(a,B)+x \\ &\preceq \mathcal{H}(A,B)+x. \end{aligned}$$

□

The result given below was proved in [33] for the real valued metric spaces. We are going to give an analogous result in C^*-valued-metric spaces which will be needed in continuation.

Lemma 5.1.11.
Let $(X,\mathbb{A},d)$ be a C^*-valued-metric space and $A,B\in CB(X)$. Assume also that the range of d is a totally ordered subset of $\mathbb{A}_+$. Then for each $a\in A$,

$$\operatorname{dist}(a,B)\preceq \mathcal{H}(A,B).$$

Proof.
Note that for each $a\in A$, we have

$$\begin{aligned} \operatorname{dist}(a,B) &\preceq \sup_{a\in A}\operatorname{dist}(a,B) \\ &\preceq \mathcal{H}(A,B). \end{aligned}$$

This completes the proof. □

Theorem 5.1.12.
Let $(X, \mathbb{A}, d)$ be a C^*-valued-metric space and assume that the range of d is a totally ordered subset of $\mathbb{A}_+$. Let the map $\mathcal{H}$ be as defined by (5.2) of Definition 5.1.8. Then $\mathcal{H}$ is a C^*-valued-metric on $CB(X)$.

Proof. Keeping in mind Definition 2.3.14 of a C^*-valued-metric on X, we proceed as follows to show that$\mathcal{H}$ is a C^*-valued-metric on $CB(X)$.

(i):

Clearly
$$\mathcal{H}(A, B) \succeq 0_{\mathbb{A}}.$$

If $\mathcal{H}(A, B) = 0_{\mathbb{A}}$, then both

$$\sup\{\operatorname{dist}(b, A) : b \in B\} = 0_{\mathbb{A}}, \tag{5.3}$$

and
$$\sup\{\operatorname{dist}(a, B) : a \in A\} = 0_{\mathbb{A}}. \tag{5.4}$$

Now from (5.3) we have

$$\operatorname{dist}(b, A) = 0_{\mathbb{A}} \text{ for all } b \in B,$$

which implies that
$$B \subseteq \overline{A}.$$

Similarly from (5.4) we get
$$A \subseteq \overline{B}.$$

Since A and B are closed we have $A = B$.

(ii) (Symmetry):

Note that the condition
$$\mathcal{H}(A, B) = \mathcal{H}(B, A)$$

follows from the fact that d is a C^*-valued-metric on X.

(iii) (Triangle Inequality):

To prove the triangular inequality assume that $A, B, C \in CB(X)$ and choose arbitrary elements $x \in \mathbb{A}_+$ and $u \in A$. There exist $v \in B$ such that

$$d(u,v) \preceq \operatorname{dist}(u,B) + \frac{1}{2}x. \tag{5.5}$$

Also there exist $w \in C$ such that

$$d(v,w) \preceq \operatorname{dist}(v,C) + \frac{1}{2}x. \tag{5.6}$$

So we have

$$\begin{aligned} \operatorname{dist}(u,C) &\preceq d(u,w) \\ &\preceq d(u,v) + d(u,w) \end{aligned}$$

Using (5.5) and (5.6) it follows that

$$\begin{aligned} \operatorname{dist}(u,C) &\preceq \operatorname{dist}(u,B) + \operatorname{dist}(v,C) + x \\ &\preceq \mathcal{H}(A,B) + \mathcal{H}(B,C) + x. \end{aligned}$$

Since u was chosen to be an arbitrary element of A, this means that

$$\sup\{\operatorname{dist}(a,C) : a \in A\} \preceq \mathcal{H}(A,B) + \mathcal{H}(B,C) + x.$$

Moreover, x is also an arbitrary element in $\mathbb{A}_+$ and we know that $\inf \mathbb{A}_+ = 0_{\mathbb{A}}$, it follows that:

$$\sup\{\operatorname{dist}(a,C) : a \in A\} \preceq \mathcal{H}(A,B) + \mathcal{H}(B,C).$$

In the similar way one can show that

$$\sup\{\operatorname{dist}(c,A) : c \in C\} \preceq \mathcal{H}(A,B) + \mathcal{H}(B,C).$$

From the above two inequalities we get

$$\mathcal{H}(A,C) \preceq \mathcal{H}(A,B) + \mathcal{H}(B,C).$$

This completes the proof. □

The following simple example illustrates the definition of the C^*-valued Hausdorff distance between two sets.

Example 5.1.13.
Let $X = [-1, 1]$, $\mathbb{A} = \mathbb{R}^2$ with the component-wise operations of addition and multiplication and the ordering $\preceq$ be as given by (3.3) of Example 3.1.2. Define $d_1 : X \times X \to \mathbb{A}$ by

$$d_1(x, y) = (|x - y|, 0).$$

Then $(X, \mathbb{A}, d_1)$ is a C^*-valued-metric space.

Let $A, B \in CB(X)$ be given by the closed intervals in X as

$$A = \left[0, \frac{1}{4}\right] \text{ and } B = \left[\frac{1}{2}, \frac{3}{4}\right]$$

Then

$$\begin{aligned}
\mathcal{H}(A, B) &= \max\{\sup\{\operatorname{dist}(b, A) : b \in B\}, \{\sup \operatorname{dist}(a, B) : a \in A\}\} \\
&= \max\left\{\left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, 0\right)\right\} \\
&= \left(\frac{1}{2}, 0\right) \in \mathbb{R}^2
\end{aligned}$$

Using C^*-Hausdorff metric on $CB(X)$ we now introduce the notion of C^*-multivalued contraction as follows.

Definition 5.1.14. (C^***-Multivalued Contractions**)
Let $(X, \mathbb{A}, d)$ be a C^*-valued-metric space and the range of d be a totally ordered subset of $\mathbb{A}_+$. Let $\mathcal{H}(A, B)$ be a C^*-valued Hausdorff metric on $CB(X)$. A mapping $T : X \to CB(X)$ is called a C^*-multivalued contraction if there exists $a \in \mathbb{A}$ with $\|a\| \leq 1$. such that

$$\mathcal{H}(Tx, Ty) \preceq a^* d(x, y) a \quad \text{for all} \quad x, y \in X. \tag{5.7}$$

The real number $\|a\|$ is called the contraction constant for the mapping T.

Example 5.1.15.
Consider again the setting of Example 5.1.13 and define $T : X \to CB(X)$ by

$$Tx = \{y : 0 \leq y \leq \frac{1}{4}x\}$$

Then

$$\begin{aligned} \mathcal{H}(Tx, Ty) &= \frac{1}{4}(|x-y|, 0) \\ &= (\frac{1}{2}, 0)d(x, y)(\frac{1}{2}, 0). \end{aligned}$$

Taking $a \preceq \left(\frac{1}{2}, 0\right)$ we have $\|a\| = \frac{1}{2}$ and

$$\mathcal{H}(Tx, Ty) \preceq a^* d(x, y)\, a.$$

Hence T is a C^*-multivalued contraction.

Motivated by the Nadler's fixed point theorem for multivalued mappings [71], we now establish a fixed point theorem for C^*-multivalued contraction in the setting of C^*-algebras.

Theorem 5.1.16.
Consider a C^*-valued-complete metric space $(X, \mathbb{A}, d)$ and assume that the range of d is a totally ordered subset of $\mathbb{A}_+$. Let the mapping $T : X \to CB(X)$ be a C^*-multivalued contraction. That is, there exists $a \in \mathbb{A}$ with $\|a\| \leq 1$. such that

$$\mathcal{H}(Tx, Ty) \preceq a^* d(x, y) a \quad \text{for all} \quad x, y \in X.$$

Then T has a fixed point.

Proof.
Let $\|a\| < 1$ be contraction constant for T and $x_0 \in X$. Consider a point $x_1 \in Tx_0$. Because both Tx_0 and Tx_1 are closed and bounded subsets of X and $x_1 \in Tx_0$, there will be a point x_2 in Tx_1 such that

$$d(x_1, x_2) \preceq \mathcal{H}(Tx_0, Tx_1) + a^* a.$$

Again, since Tx_1 and Tx_2 are closed and bounded subsets of X and x_2 lies in Tx_1, there will be a point x_3 in the subset Tx_2 which satisfies

$$d(x_2, x_3) \preceq \mathcal{H}(Tx_1, Tx_2) + a^{*2}a^2.$$

Proceeding in this way we obtain a sequence $\{x_i\}_{i=1}^{\infty}$ of points of X such that x_{i+1} lies in Tx_i and

$$d(x_i, x_{i+1}) \preceq \mathcal{H}(Tx_{i-1}, Tx_i) + (a^*)^i a^i$$

for all $i \geq 1$.

We note that for all $i \geq 1$,

$$\begin{aligned}
d(x_i, x_{i+1}) &\preceq \mathcal{H}(Tx_{i-1}, Tx_i) + (a^*)^i a^i \\
&\preceq (a^*)d(x_{i-1}, x_i)a + (a^*)^i a^i \\
&\preceq a^*[\mathcal{H}(Tx_{i-2}, Tx_{i-1}) + (a^*)^{i-1}a^{i-1}]a + (a^*)^i a^i \\
&= a^*[\mathcal{H}(Tx_{i-2}, Tx_{i-1}]a + 2(a^*)^i a^i \\
&\preceq \cdots a^{*i} d(x_0, x_1)a^i + i(a^*)^i a^i
\end{aligned}$$

Hence

$$\begin{aligned}
&d(x_i\, , x_{i+j}) \\
&\quad \preceq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + \cdots + d(x_{i+j-1}, x_{i+j}) \\
&\quad \preceq a^{*i} d(x_0, x_1)a^i + i(a^*)^i a^i + (a^*)^{i+1} d(x_0, x_1)a^{i+1} + (i+1)(a^*)^{i+1}a^{i+1} + \\
&\quad \cdots (a^*)^{i+j-1} d(x_0, x_1)a^{i+j-1} + (i+j-1)(a^*)^{i+j-1}a^{i+j-1} \\
&\quad = \sum_{n=i}^{i+j-1} (a^*)^n d(x_0, x_1)a^n + \sum_{n=i}^{i+j-1} ((a^*)^n a^n)
\end{aligned}$$

Using 3.7 of Lemma 3.1.12, we see that for all $i, j \geq 1$,

$$d(x_i, x_{i+j}) \longrightarrow 0_{\mathbb{A}} \text{ as } j \longrightarrow \infty$$

It follows that the sequence $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence in X with respect to $\mathbb{A}$.

Since $(X, \mathbb{A}, d)$ is complete with respect to $\mathbb{A}$, the sequence $\{x_i\}$ will converge to some x_0 in X. Also,

$$\mathcal{H}(Tx_i, Tx_0) \preceq a^* d(x_i, x_0) a.$$

Therefore, the sequence $\{Tx_i\}_{i=1}^{\infty}$ converges to Tx_0. Also x_i lies in Tx_{i-1} for all i, and

$$\lim_{i \to \infty} \operatorname{dist}(x_i, Tx_0) = 0_{\mathbb{A}}$$

and since Tx_0 is closed, it follows that $x_0 \in Tx_0$. $\square$

Remark 5.1.17. If we take $\mathbb{A} = \mathbb{R}$ then our result coincides with the result proved by [71].

Conclusion

Recently, Kadelburg and Radenovic [52] and Alsulami et al. [3] noted that the fixed point results in C^*-algebra valued-metric spaces can be obtained from the corresponding results in complete metric spaces using the machinery of C^*-algebra. By comparing there findings with the proofs of theorems given by Ma et al., we observe that the proofs given by [62] can be shorten by using the argument provided by Kadelburg and Radenovic [52] and Alsulami et al [3]. But they have used the same results from C^*-algebra as used by [62] Therefore, we conclude that the approach adopted by them is essentially the same as that of Ma et al. The only difference seems to us is that they have used the existing fixed point results to shorten their proofs whereas Ma et al. have given the detailed proofs.

In this thesis, we have proved some fixed point theorems in the setting of C^*-valued-metric spaces following the approach adopted by Ma et al.

Moreover, we have noted that the notion of C^*-valued-metric space is different from cone metric space as mentioned in [96] as follows: "Let E be a real Banach space. A cone P in E defines a partial ordering in E as follows: let $x, y \in E$

we say $x \preceq y$ if $y - x \in P$. Using this partial ordering Huang and Zhang [50] introduced the notion of a cone metric space. A cone metric on a nonempty set X is a mapping $d_c : X \times X \to E$ satisfying: (i) $d_c(x, y) > 0$ for all $x, y \in X$ and $d_c(x, y) = 0$ if and only if $x = y$; (ii) $d_c(x, y) = d_c(y, x)$ for all $x, y \in X$; (iii) $d_c(x, z) \leq d_c(x, y) + d_c(y, z)$ for all $x, y, z \in X$. In fact this notion is not new and was initially defined by Kantorovich [57] as a K-metric space [51, 57]. Huang and Zhang [50] called a mapping $f : X \to X$ a cone contraction if it satisfies following condition.

$$d_c(fx, fy) \leq kd_c(x, y) \ \forall \ x, y \in X \text{ for some } k \in (0, 1). \tag{5.8}$$

Then they generalized the Banach contraction principle in the context of cone metric spaces [50, Theorem 1]. Note that $d_c(x, y)$ is an element of the Banach space E and the right hand side of (5.8) is defined, since E is a real Banach space. The set of positive elements in a C^*-algebra forms a positive cone in the C^*-algebra but the underlying vector space is not a real vector space, in general. Therefore, the notion of a C^*-valued-metric space seems is general than the notion of cone metric space. For example if we consider the set $\mathbb{A}$ of all 2×2 matrices having entries from complex numbers, then $\mathbb{A}$ is a vector space over the field of complex numbers. Also, $\mathbb{A}$ is a C^*-algebra with Euclidean norm. A mapping $T : X \to X$ is said to be a C^*-valued contraction mapping on X, by Ma et al., (Definition 2.4.9) if there exist an A in a C^*-algebra $\mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx, Ty) \preceq A^* d(x, y) A, \quad \text{for all } x, y \in X. \tag{5.9}$$

Observe that the right hand side of (5.9) is defined because $\mathbb{A}$ is an algebra, not necessarily real. Also, observe that, it is not necessary that one can define an involution "$*$" on a normed space. Thus it seems to be difficult that the inequality (5.9) can be reduce to the inequality (5.8). Further note that the proof of the main result by Ma et al. [62] depends on machinery of C^* algebras. Thus we conclude that the main results of Ma et al. and ours may not follow from the corresponding results of cone metric spaces."

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