CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Robust Control Of Under-actuated Systems via Sliding Modes

by

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A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy

in the

Faculty of Engineering
Department of Electrical Engineering

2019

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ISLAMABAD

2019

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Dedicated to my wife, parents and teachers

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CERTIFICATE OF APPROVAL

This is to certify that the research work presented in the thesis, entitled "**Robust Control of Under-actuated Systems via Sliding Modes**" was conducted under the supervision of **Dr. Fazal ur Rehman**. No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the **Department of Electrical Engineering, Capital University of Science and Technology** in partial fulfillment of the requirements for the degree of Doctor in Philosophy in the field of **Electrical Engineering**. The open defence of the thesis was conducted on **May 3, 2019.**

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List of Publications

It is certified that following publication(s) have been made out of the research work that has been carried out for this thesis:-

Journal Articles

1. **S. U. Din**, Q. Khan, F. U. Rehman, and R. Akmeliawati, "Robust Control of Underactuated Systems: Higher Order Integral Sliding Mode Approach," *Mathematical Problems in Engineering*, Article ID 5641478, 11 pages, 2016. DOI: 10.1155/2016/564147. **(Impact Factor. 1.1)**

2. **S. U. Din**, Q. Khan, F. U. Rehman, and R. Akmeliawati, "A Comparative Experimental Study of Robust Sliding Mode Control Strategies for Underactuated Systems," *IEEE Access*, vol. 6, pp. 1927-1939, 2018. DOI: 10.1109/ACCESS.2017.2780889. **(Impact Factor. 3.5)**

3. **S. U. Din**, F. U. Rehman, and Q. Khan, "Smooth Super-twisting Sliding Mode Control for the Class of Underactuated Systems," *Plos One*, vol. 13(10), pp. 1-21, 2018. DOI: 10.1371/journal.pone.0203667. **(Impact Factor. 2.7)**.

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Acknowledgements

First of all, I would like to thank Almighty Allah, who gave me strength and wisdom and courage in every aspect of life. Thanks and regards to the all glory prophet Mohammad (PBUH), for his enlightening teachings, that makes me focused and concentrated.

I would like to express my sincere gratitude to my supervisor Dr. Fazal-ur-Rehman for his immense support, guidance and encouragement throughout my doctoral studies. His kind effort highly contributed in enhancing my knowledge and research enthusiasm. His research knowledge and experience directed me on the right direction to complete my studies on time. It has been a great honor for me to work under his supervision.

I am also thankful to my co-supervisor Dr. Qudrat Khan. His guidance in finding my research direction will always be unforgettable for me.

I am also highly obliged to all my teachers Dr. Aamer Iqbal Bhatti, Dr. Raza Samar and Dr. Noor Muhammad Khan, for providing me necessary background knowledge.

I am also thankful to my group members, Waseem Abbasi, Ibrahim Shah and Sarfaraz Ahmed for there continues support in hard times during the study. Special thanks to Mirza Yasir Masood and Saif ur Rehman for being there, when required. I also want to pay my gratitude to my dearest friend Dr. Imran Khan for being so helpful.

I would like to thank the Higher Education Commission (HEC), Pakistan for its financial support for completion of PhD degree. I would also like to thank The University of Lahore, Islamabad Campus for providing me Control Lab for my work.

In the last, I would like to pay special thanks to my parents, brother, sister, my wife and my children's Inayah and AbdurRafay, for their support and patience during my studies.

Abstract

Underactuated nonlinear systems are always equipped with less number of actuators than the degree of freedom. This feature offers certain benefits like reduction in weight and minimum energy usage. Majority of the robotic systems (including aerial, underwater and ground robotics) are found to be underactuated in nature. Therefore, research in such system is still quite demanding and challenging. It is also worthy to mention that the underactuation phenomenon, do not allow the direct design of control input as practiced in fully actuated systems. The two decades have witnessed many control methodologies which include feedback linearization, energy-based, back-stepping, fuzzy logics and sliding mode control. However, majority of these techniques lags behind in the robust stabilization of this class except sliding mode oriented techniques. An extensive simulation study of the underactuated system is carried out in the existing literature while considering the examples of translational oscillator with a rotational actuator (TORA), flexible robots, pendulums and surface vessels.

In this thesis a simulation as well as experimental study is carried out for a class of underactuated systems. The nonlinear model, of the underactuated systems, is treated generally. The dynamics are either transformed into an input output form and then an integral manifold is devised for the control design purpose or an integral manifold is defined directly for the concerned class. Having defined the integral manifolds discontinuous control laws are designed which are capable to maintain sliding mode from the very beginning. The closed loop stability of these systems is presented in an impressive way. The effectiveness and demand of the designed control laws are proved in term of simulation and experimental results of a ball and beam system. In addition, a comparative experimental study is also performed between three generations of sliding mode control, which includes the conventional first order sliding mode control (FOSMC), second order sliding mode (SOSMC), fast terminal sliding mode (FTSMC), and integral sliding mode (ISMC). The comparative study takes into account certain features like tracking

performance, i.e., settling time, overshoots, robustness enhancement, chattering reduction, sliding mode convergences and control efforts.

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Abbreviations

AI	Artificial Intelligence
FC	Fuzzy Control
FOSMC	First Order Sliding Mode Control
FTSMC	Fast Terminal Sliding Mode Control
HOSM	Higher Order Sliding Mode
HOSMC	Higher Order Sliding Mode Control
IDA	Interconnection and Damping Assignment
IDA-PBC	Interconnection and Damping Assignment Passivity Based Control
ISM	Integral Sliding Mode
ISMC	Integral Sliding Mode Control
MEMS	Micro-electromechanical Systems
MIMO	Multi Input Multi Output
PBC	Passivity Based Control
PFL	Partial Feedback Linearization
RISMC	Robust Integral Sliding Mode Control
RTA	Real Twisting Algorithm
SMC	Sliding Mode Control
SOSMC	Second Order Sliding Mode Control
STA	Super Twisting Algorithm
SSTA	Smooth Super Twisting Algorithm
TSMC	Terminal Sliding Mode Control
UAS	Underactuated Systems
UMS	Underactuated Mechanical Systems
VLSI	Very Large Scale Integration

VSS	Variable Structure System
VSC	Variable Structure Control

Symbols

A	System matrix	–
B	Input matrix	–
$\beta(t)$	Lever angle	deg
C_m	Motor torque constant	Nm/A
C_b	Back *emf* value	V/(rad/s)
C_g	Ratio of gear	–
d	Radius of arm connected to servo motor	m
g	Gravitational acceleration	m/s^2
J	Energy utilization	Joules
J_1	Beam moment of inertia	kgm^2
J_m	Motor moment of inertia	Nm/(rad/s^2)
K	Gain of controller	–
k_i	i^{th} gain of controller	–
L, l	Length	meters
M, m	Mass	Grams
M_p	Overshoot	–
r	Sliding order	–
R_m	Motor armature resistance	Ω
$r(t)$	Position of the ball on beam	cm
r_d	Desired position	cm
σ, S, s	Sliding surface	–
t_s	Settling time	Seconds
t_r	Rise time	Seconds

T_i	Derived parameters	–
t	Time	Seconds
$V, V(t, x)$	Lyapunov function	–
v_{in}	Input voltage	V
x	State vector	–
x_0	Initial conditions	–
y	Output of interest	–
z	Integral term	–
ρ	System input	–
ρ_{eq}	Equivalent control component	–
ρ_{sw}	Discontinuous/switching control component	–
$\xi(x, t)$	Perturbation	–
ξ_i	Transformed states	–
ΔG_m	Matched model uncertainties	–

Chapter 1

Introduction

After the evolution of mankind on earth, they had started to invent things for their ease. This desire of ease opened the door towards a never-ending cycle of research, inventions, and discoveries. For their comfort and ease, mankind has discovered fire and invent the wheel in the early stone-age. After the invention of the wheel around 3500 BC, obligations raised regarding velocity/momentum of the wheel. This urge about speed control of wheel flourishes the concept of research, which is still continuing. Everyday sun shines with the new ideas, innovation, discoveries, and inventions.

As humans are very ambitious to control their surrounding in accordance with their ease and comfort, so it makes control system an essential part of their life. In addition to this Micro-electromechanical system (MEMS) and Very Large-Scale Integration (VLSI) makes their dreams to come true. The control system is somehow a part of every necessary equipment which we are utilizing in our daily life, whether it belongs to electrical, mechanical, chemical or even biological system. It is used to obtain desired goals in hardware or in a software way. The most important thing for control engineers/researchers while outlining the control framework, is its mathematical model. The precision of mathematical model lead towards the precise control of the desired systems, but unfortunately, the mathematical model is not very accurate towards the real system. Now the role of control engineers/researchers is involved to design a control system, which able to achieve robustness along with considerable effective performance in presence of model imprecisions and external disturbances.

The classical control theory introduces in 1930's, and it became a part of engineering discipline in late 1950's. Control engineer design and control the systems, using graphical solutions which includes Root locus, Bode plot, and Nyquist plot. All these methods belong to frequency domain analysis, in addition to this it utilizes the open loop transfer function of the system and determines the response of the closed-loop system. These classical methods are complicated to implement on complex non-linear systems and don't have the capability to handle the irrational functions, e.g. (delays, pade approximations). If we derive the model with some approximations from the original non-linear model, then the controller designed on such model also has some limitations regarding control of a real system. In result of such constraints, modern control theory comes to place.

Modern control techniques use state space representation for control systems and develop the relationship between inputs and outputs of the system via Ordinary Differential Equation (ODE). This method is comparatively easy to analyze MIMO and complex systems in view of controller design of nonlinear systems. Moreover, the design of a nonlinear controller for a non-linear system eliminate the imprecision produced by the approximation of a non-linear model.

When it comes to underactuated systems,studies can be traced back to the existing and last decade. Underactuated systems by definition contain less no of control input/actuators as compared to the degree of freedom [1]. The feature of underactuation makes their control distinct from other nonlinear systems (also called fully-actuated systems). Usually, the feature of under-actuation raises due to following four reasons [2],

- **System Dynamics:** Dynamics of the system may also be the cause of the rise of under-actuation phenomenon, usually seen in aerial and underwater vehicles, (locomotive systems independent of wheels).

- **By Design:** It can be deliberately introduced for reduction of cost and for getting more practical advantages like reduction of weights in space/underwater vehicles, humanoids manipulators.
- **Actuator Malfunction:** Under-actuation may raise in case of actuator failure, e.g., in case of collision with another object [3].
- **Obtaining Low-order Nonlinear Systems:** Feature of under-actuation can artificially be imposed to create low order- nonlinear systems for the purpose of gaining insight control of higher order underactuated systems e.g., a ball on a beam systems [4], cart-pole systems [5], Translational Oscillator with Rotational Actuator (TORA) [6].

This class finds fascinating applications in humanoids, aerospace systems and underwater vehicles, mobile and locomotive systems. In order to operate such kind of systems autonomously, very sophisticated control techniques are required. The functioning of any system rely on input signal known as control input generated by following some control design strategy or algorithm. There are many methodologies by which ones can generate these kinds of algorithms in control theory, but one of them is the Sliding Mode Control (SMC) technique which is known for its simpler design methodology and robust nature against uncertainty/disturbances and applicability to both linear and nonlinear systems.

The fundamental thought behind this strategy is to invoke sliding mode in the system. The occurrence of the sliding mode is associated with a constraint, termed as a sliding manifold. The beauty of sliding mode is that it guarantees the robustness against specific class, regarding parametric uncertainties, external disturbances and un-modeled dynamics [7-8]. Although, it suffers from the high-frequency oscillations across the manifold (known as chattering phenomena). This phenomenon is very dangerous for the actuators health; it may lead towards system failure in a severe case. The aforementioned control strategy may appear in many variants, widely addressed in [9]. As these variants belong to variable structure control, therefore, these variants have some common steps in design procedure.

Conventionally, SMC can be implemented in two phases, reaching phase and sliding phase. As illustrated by the name, all the states of the system converge to the sliding manifold in reaching phase. In sliding phase, these states remain on sliding manifold as far as the equilibrium position is not achieved. Reduced order dynamics is the extra advantage of the sliding mode control, in addition, its play momentous role regarding robustness. The states of the system become insensitive to disturbances/uncertainties, once they reach sliding manifold.

Sliding mode shows very appealing result in real applications; the only problem exists, is the occurrence of high-frequency switching (also calls chattering) when the sliding mode established. Researchers have proposed different methods to overcome chattering problem. More concerns regarding SMC includes, lack of accuracy and considerable trade-off among robustness and chattering. In the current era, control researchers are vigorously working towards chattering reduction, performance improvement, and robustness enhancement.

The motivation of this dissertation is being presented in the upcoming section.

1.1 Motivation for the Work

In the current era, humanoids/robots are broadly employed in industries to reduced labour force, for economical manufacturing. Quite sophisticated mechatronic and micro-electromechanical systems are designed to fulfill the needs of industry and healthcare and is highly appreciated by the society [10]. All these tasks are mainly performed via systems having fewer inputs then the outputs called underactuated systems (UAS) The research on the UAS is quite significant and it put forward certain challenges [11], For example:

- On the basis of current/existing control approaches, can UAS be controlled better?
- Does any novel UAS exis? that may be able to provide a solution to our issues in real?

Aforesaid questions places this class of nonlinear system still an open area of research. The main task in UAS systems reduced to robust stabilization and robust tracking with acceptable performance. This objective is generally accomplished via feedback control of the system outputs. It is noticeable that the control of this class (UAS) is entirely different from the fully actuated systems where outputs are monitored via equal number of inputs.

The control of underactuated frameworks are entirely different from the other nonlinear plants where the systems operate with the same number of inputs and outputs (so-called fully actuated systems). The control problem of fully actuated systems is not considered to be a big issue, reason behind that is there are matured enough control techniques available to perform exact feedback linearization on such systems directly, and later on their control can be designed by using any frequency domain analysis or by applying simple pole placement methodology. On the other hand, in underactuated systems,due to its underactuation phenomenon control does not apply directly as in fully actuated systems.The reason of underactuation may be the system dynamics (by nature like ariel and underwater vehicles), or it can be introduced deliberately to attain certain practical advantages like [12]:

- actuator malfunction/failure
- low complexity level
- minimal cost
- lightweight
- minimal power utilization/consumption
- low damage risk/cost (in case of an accident)

Practical applications of the underactuated system include [12]:

- In robotics; fixed and mobile robot, flexible link joints.

- In ariel and underwater vehicles; spacecraft, space exploration system, aircraft, helicoptors, surface vessels, ships.
- In industry; transportation and object manipulator system.
- In education institutes for research and innovation; ball and beam system, TORA, inertial wheel pendulum, acrobot, pendubot, overhead crane, cart-pole system.

Due to their complex dynamics and highly nonlinear nature, a system by system approach is proposed by the researchers regarding the control design of the underactuated systems. Furthermore, external disturbance in combination with model uncertainties becomes a major issue in the control of underactuated systems. To fulfill the aforesaid requirements of the underactuated framework, SMC variants are plugged-in. The trend of using SMC variants for the observation and control purpose, drive the researchers to examine/analyze these algorithms for all possible improvement. Regarding SMC, chattering problem can be overcome to some extent by applying Higher Order Sliding Mode Control (HOSMC), but HOSM is very sensitive towards unmodeled fast dynamics. On such situations, where unmodeled fast dynamics raises, the minute value of chattering may lead towards total chaos (in terms of system instability). Secondly, it is difficult to implement from a practical point of view. Therefore, the aforesaid discussion stimulates to intend a control design approach that may able to provide required robustness with minimal chattering and enhanced performance. The approach might have the capacity to meet over three issues in an optimal sense. In this paragraph, optimality is meant to acquire acceptable performance with substantially reduced chattering and improved robustness at a time. In this research, following assumptions are considered to be fulfilled.

- The system output must be measurable.
- The uncertainties/disturbances(vanishing or non-vanishing) are bounded.
- The system parameters are precise/measurable.

- The relative degree is one with respect to the sliding manifold.

In the upcoming section, research objective and scope of this dissertation is discussed.

1.2 Research Objective and Scope

The above discussion and analysis clarifies the added advantages coming from the under actuation phenomenon along with the theoretical importance and practical applications of underactuated systems. However, their realization is limited to practical systems because such benefits come with a higher cost, in terms of complex control design coupled with highly nonlinear dynamical system due to underactuation. In addition to this, the existence of probability towards mismatch between real plant and mathematical model in the presence disturbances makes the control design even more complex and challenging. Therefore designing robust nonlinear control in the presence of aforesaid constraints (mismatched/uncertain mathematical model, matched and unmatched disturbances, which are more common in real-world scenario) is a considerable imperative control problem.

Sliding mode control [13], is the only robust design technique which has the capability to provide robustness and cater for unknown internal and external disturbance in the presence of model uncertainties for nonlinear systems. Other approaches developed for such systems include partial feedback linearization [14-15], passivity-based approach [16], backstepping [17], IDA-PBC [18], fuzzy control [19] and optimal control [20-22], these approaches have certain limitations but most important of them is the lack of robustness. Moreover, conventional SMC [23-25] and their higher order variants [26-27], suffers the high-frequency oscillations (known as chattering) which makes practical applicability of such controllers nearly impossible.

The research objective in this work is to find robust control design for the class of underactuated system via sliding mode. The proposed controller should be, robust enough from the very beginning, suppress chattering, improve performance and can be applicable to the underactuated system that can be converted in to canonical form. The generic model of the underactuated systems is considered for the said propose includes the systems like ball and beam, cart-pole system, overhead crane, pendubot, TORA, acrobot. Finally, the framework is practically implemented and tested on the ball and beam system being a benchmark underactuated systems.

The scope of this research can be further extended in future, which includes the addition of such systems/models which are not include in considered class due to certain design limitations.

In the upcoming section, main research contributions are listed, in the form of control law development (for robustness enhancement, performance improvement, and chattering attenuation).

1.3 Research Contribution

The contributions made in this manuscript are both theoretical and experimental in its nature. A robust control strategy is proposed for UAS. To design the control strategy, the system needs to be in the specific formate, i.e., input-output form. To get the system into this format, some transformations will be needed. Once the system is obtained in the said format an Integral Sliding Mode Control (ISMC) law is proposed for the tracking purpose to ensure the robust tracking performance. The closed loop stability is proved, and the designed control law is experimentally tested on a ball and beam system. In addition, a comparative experimental studies of the conventional First Order Sliding Mode Control (FOSMC), Second Order Sliding Mode (SOSMC), Fast Terminal Sliding Mode Control (FTSMC), and Integral Sliding Mode Control (ISMC) is also performed. The comparative study take into account certain features like tracking performance i.e., settling time, overshoots, robustness enhancement, chattering reduction, sliding mode convergences and control efforts., These contributions are quite significant and upto the mark. The following contribution, in term of research papers, is made in this thesis.

- Robust Control of Underactuated System: Higher Order Integral Sliding Mode Control [28].
- A Comparative Experimental Study of Robust Sliding Mode Control Strategies for Underactuated Systems [29].

1.4 Thesis Structure

A short outline of the substance contained chapter wise in this dissertation is given below.

Chapter 1 presents the introduction of this dissertation which includes a brief discussion of sliding mode control literature, motivation for this work, research contribution and thesis organization.

Chapter 2 explores the historical evolution of the underactuated systems. The central theme of this chapter is to provide an overview regarding remarkable work done on underactuated systems with respect to control. In addition, it also reveals the step by step progression takes place in SMC along with its merits and demerits.

Chapter 3 reflects some necessary mathematical concepts, which seems to be necessary for the better understanding of future chapters. This chapter effectively presents the fundamental theory regarding SMC, Second Order Sliding Mode Control (specifically, super-twisting sliding mode control) SOSMC (STW), Smooth Sliding Mode Control (SSTW), ISMC and FTSMC.

Chapter 4 contains the many contributions to this thesis, starting from the development of a controllable canonical form of a given nonlinear model of the class of underactuated nonlinear systems. The main claim of this chapter is the development of the Robust Integral Sliding Mode Control (RISMC) protocol for the class of underactuated nonlinear system. The stability analysis is carried out via Lyapunov's approach. In addition to this, RISMC algorithm is practically implemented on Ball and Beam system (underactuated nonlinear uncertain) system.

Chapter 5 contains the extended form of the contribution presented in chapter 4. In this chapter, comprehensive comparative analysis in term of settling time, overshoots, robustness enhancement, chattering reduction, sliding mode convergences and control efforts is performed with five sliding mode control strategies i.e., conventional FOSMC, SOSMC, FTSMC, ISMC. Aforementioned control techniques are implemented on a Laboratory benchmark to observe the characteristics of each control technique very closely.

Chapter 6 conclusion of this dissertation is presented in this chapter along with some future directions.

1.5 Summary

This chapter has provided an overview of this dissertation. In the next chapter literature survey with respect to the underactuated systems are presented.

Chapter 2

Literature Survey

This chapter illuminates the historical perspective of different control strategies and algorithms exercised regarding the control of underactuated systems by the research community. The maturation from conventional control strategies to SMC initiated an everlasting research cycle which diversified itself toward different variants of SMC like, HOSMC, Terminal SMC, ISMC, FTSMC, etc.

The control design of the underactuated systems remains in one of the active fields for control engineers/ researchers in the existing and the last decade. Underactuated system are the systems with fewer actuators (i.e., controls) then configuration variables [5], this property makes it quite unique from other systems. These systems are used in order to have a minimum weight, cost, and energy usage while still retaining the key features of the underactuation. Note that, in case of fully actuated systems, there exists a broad range of design techniques in order to improve performance and robustness. These include adaptive control, optimal control, feedback linearization, and passivity-based control strategies, etc.

However, it may be difficult to apply such techniques in a large class of underactuated systems because sometimes these systems are not linearizable using smooth feedback [30] also due to the existence of unstable hidden modes in some such systems. Brockett [31] necessary condition for the hold of stable, smooth feedback law is also not satisfied by the majority of under-actuated systems. This modern era has a wide application of these underactuated systems which may be able to operate manually/automatically.

In recent years many control methodologies have applied for stabilizing the underactuated systems to full-fill our needs, which includes: feedback linearization technique, energy-based techniques, back-stepping, fuzzy logic and sliding mode control techniques.

2.1 Partial Feedback Linearization (PFL)

PFL (Partial feedback linearization technique [14-15]) provides a natural global change of coordinates that transforms the system into strict feedback form. Following two PFL approaches are used for underactuated systems namely,

- Collocated PFL
- Non-collocated PFL

2.1.1 Collocated PFL

The linearization scheme globally transform all UAS of some kind to fully actuated systems is laid out by collocated PFL approach. Application of this control technique includes the control of acrobot [31], capsule system [32], double pendulum cart [33], three link- pendulum [34], pendulum driven cart [35] and cart-pole system [36].

Example

For better explanation of PFL approach, Lagrangian based model of inertial wheel pendulum (IWP) is presented in equation (2.1), [15].

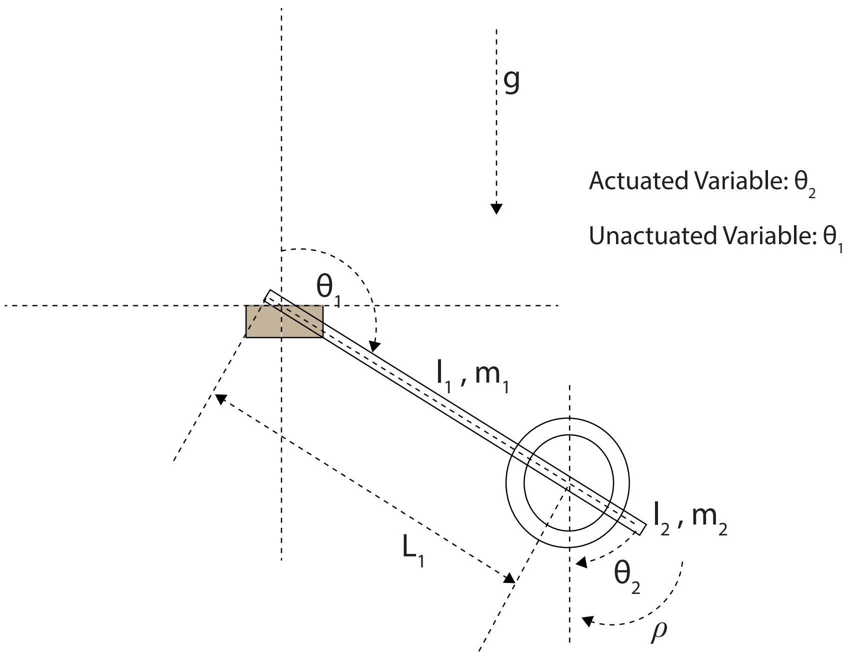

FIGURE 2.1: Schematic representation of IWP (Inertia Wheel Pendulum [15]).

$$\begin{cases} m_{11}(q)\ddot{q}_1 + m_{12}(q)\ddot{q}_2 + h_1(q,p) = 0 \\ m_{21}(q)\ddot{q}_1 + m_{22}(q)\ddot{q}_2 + h_2(q,p) = 0 \end{cases} \tag{2.1}$$

From above equation, one may write equation (2.2)

$$\ddot{q}_1 = -m_{11}^{-1}(q)\{m_{12}(q)\ddot{q}_2 + h_1(q,p)\} \tag{2.2}$$

By replacing the value of $\ddot{q}_1$ by the above equation, second actuated dynamical equation of (2.1) can be rewritten as:

$$\left(m_{22}(q) - m_{21}(q)m_{11}^{-1}(q)m_{12}(q)\right)\ddot{q}_2 + \left(h_2(q,p) - m_{21}m_{11}^{-1}(q)h_1(q,p)\right) = \rho \tag{2.3}$$

By applying the control input displayed in equation (2.4), above equation can be linearizeable.

$$\rho = \left(m_{22}(q) - m_{21}(q)m_{11}^{-1}(q)m_{12}(q)\right)\left(\tau + \left(h_2(q,p) - m_{21}m_{11}^{-1}(q)h_1(q,p)\right)\right) \tag{2.4}$$

In above equation (2.4), new control input is represented by τ. Therefore, equation (2.1) can be rewritten as follows:

$$\begin{cases} m_{11}(q)\ddot{q}_1 + h_1(q,p) = -m_{12}(q)\tau \\ \ddot{q}_2 = \tau \end{cases} \tag{2.5}$$

More simplified version of above equation (2.5) can be shown in following equation (2.6), shown below:

$$\begin{cases} \dot{q}_1 = p_1 \\ \dot{p}_1 = f(p,q) + g(q)\tau \\ \dot{q}_2 = p_2\dot{p}_2 = \tau \end{cases} \tag{2.6}$$

where

$$\begin{cases} f(q,p) = -m_{11}^{-1}(q)h_1(q,p) \\ g(q) = -m_{11}^{-1}(q)m_{12}(q) \end{cases} \tag{2.7}$$

It is worthy to mention, in actual collocated linearization technique, linearizes the actuated degree of freedom, in addition decouples it from the unactuated Degrees of Freedom (DoF).

2.1.2 Non-Collocated PFL

Non-collocated PFL presents a linearization procedure for the un-actuated configuration variables, this procedure also valid for a specific class of UAS [11]. Olfati-Saber [37] also works on non-collocated PFL for UAS with respect to symmetry property. Applications of aforesaid technique includes rotating pendulum [37], surface vessel [38], pendubot [39] and one link robot [40].

Example

By continuing above example (system shown in equation 2.1) for non-collocated PFL approach [15], if one wants to linearize q_1, the control input is selected as

$$\vartheta = -m_{11}^{-1}(h_1 + m_{12})\tau \tag{2.8}$$

That becomes

$$\begin{cases} \ddot{q}_1 = \vartheta \\ \ddot{q}_2 = -m_{12}^{-1}(q)(m_{11}(q) - h_1(q,p)) \end{cases} \tag{2.9}$$

Therefore, state space model of IWP, can be shown as

$$\begin{cases} \dot{q}_1 = p_1 \\ \dot{p}_1 = \vartheta \\ \dot{q}_2 = p_2 \\ \dot{p}_2 = \frac{\vartheta - f}{g} \end{cases} \tag{2.10}$$

From the mathematical model displayed in (2.10) that, non-collocated linearization yields decoupling of unactuated DoF. Aforesaid techniques are inapplicable on flat underactuated systems due to existence of non-zero gravity term ($G_u(q) \neq 0$).

2.2 Passivity Based Control (PBC)

The idea behind this methodology is to regulate the energy of the system in accordance with the desired equilibrium state. Passivity is the imperative aspect of underactuated systems. Feedback control law invariably exits for

$$\dot{E}(q, \dot{q}) \leq 0 \tag{2.11}$$

For set point regulation problems mostly used technique is PBC, this strategy is commonly applied on two link manipulators [16], biped robot [41], two serial pendulum cart [33], TORA [6], rotating pendulum [42], two parallel pendulum cart [43] and cart-pole [44]. Narrow range of application and only applicability to systems having less than two relative degrees are the limitations/drawbacks of passivity based approach [11].

2.3 Backstepping

To counter the limitation of passivity-based control (PBC), the back-stepping technique is proposed to transform the system into recursive form in which PBC can easily be applied. The principle of back stepping is displayed by following figure [17]:

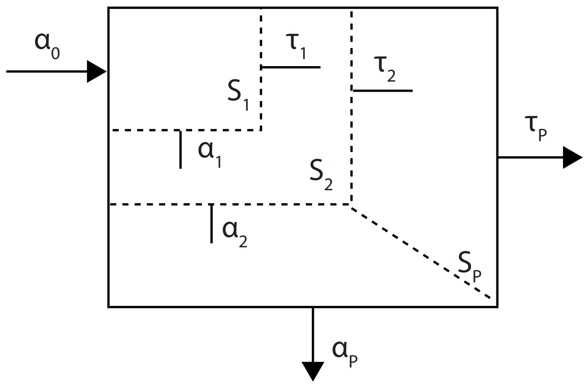

FIGURE 2.2: Scheme of backstepping [17].

In the Fig. 2.2, virtual system are represented by S_i $(i = 1{,}2{,}\cdots p)$ with r(relative degree) = 1. In back-stepping technique actual system is finished as a last member of sequence. For each S_i, r trims to one by choosing signal α_{i-1} as virtual input and control law is obtained $u = \alpha_p$.

This approach is considered to be quite impressive for global stabilization of low degree of freedom underactuated systems [11]. However, computational complexities increase with the increase in the DOF. In addition, the applications of the back-stepping technique towards practical systems are unrealistic, due to increase in complexity level as DOF increases [11, 29]. This technique is applied to control surface vessel [45-46] and VTOL aircraft [47-48].

2.4 IDA-PBC and Controlled Lagrangian Methods

Generally, two approaches are followed to illustrate the behavior of dynamical systems (namely Hamilition approach and Euler-Lagrange approach). As there are two passivity based methods for the control of underactuated systems: IDA-PBC (interconnection and damping assignment passivity-based control) [18] and controlled Lagrangian [49-50]. Both Hamilition/Lagrangian based passivity schemes based on two phases:

- Shape the Hamilition/Lagrangian to the required form, along convenient equilibrium states with desirable structural feature via control input and
- next is to implant damping in the system, to make sure the passivity of the system.

Injection of damping term leads us to a new world of control by discovering sliding modes control.

2.5 Fuzzy Control (FC)

Generally, disproportion has been seen in human and machine control. It exists due to the human factor, which leads toward imprecise, uncertain and fuzzy but the fusion of machine with computer portrays very fascinating and accurate results. An fuzzy control (FC), make machines more intelligent and make them capable of taking decisions in a fuzzy manner like a human.

This technique is proposed by Zadeh [19], it is premeditated to be a non-mathematical approach used for underactuated systems that are not adequately modeled or well defined [51]. Takagi-Sugeno modeled based approach is very famous regarding FC. Figure 2.3 shows fuzzy control scheme [11]. From set-point regulation and tracking point of view, FC applied to underactuated systems referred in [51-54].

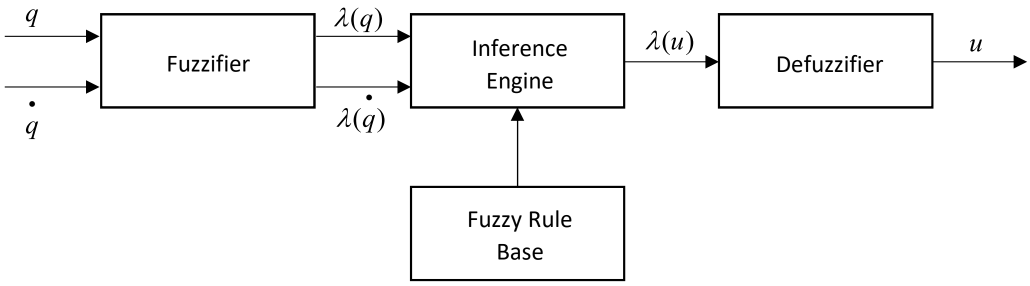

FIGURE 2.3: Scheme of Fuzzy Control [11].

2.6 Optimal Control

The objective of optimal control is to find control law having the ability to reduce (or maximize) cost function.

$$J = \int f(q(t), \dot{q}(t), u(t)).dt \tag{2.12}$$

Optimization leads us to two type of control problems, energy-optimal problem, and time-optimal problem. Relevant work concerning energy optimization and time optimization are reported in [20-21], [22] and [55] respectively.

Truthfully saying, aforesaid control scheme stills considered as control problem rather than an independent control methodology [11].

2.7 Sliding Mode Control

Uncertainty is a pervasive and still stubborn issue for the control of UAS. As there remains a gap between practical systems and there theoretical models, which leads us to model uncertainties, unmodeled dynamics and parameter variation/uncertainties. In earlier, to overcome the issue of uncertainty techniques namely, robust control [13] and adaptive control [56] were used. Owing to the nature of aforesaid two strategies, robust control is only applicable towards small uncertainties [57], on the other hand, adaptive control is able to cater broad spectrum of parametric uncertainties but sensitive towards unstructured uncertainties [11].

To counter these constraints, variable structure controller known as SMC gains extensive recognition from engineers/researchers in recent years. SMC provides remarkable system performance against disturbance rejection, model imperfection, and uncertainties. As control law designed in the result of SMC is discontinues in nature, control input enforce the system to slides along a pre-defined surface, according the system produces the desired behavior via confining its state to the surface.

SMC strategy is found to be very successful with respect to underactuated systems, their applications can be found in inverted pendulum [5], surface vessel [23-24], helicopter [25], ball and beam [28], satellite [58], overhead crane [59], underactuated fuel cell system [60], underactuated biped robot [61] and TORA [62]. Meanwhile, some researchers/engineers also devoted themselves to develop a universal SMC for the underactuated systems [63-65].

Although, SMC grew very rapidly from last two decades, but it suffers from a high oscillation phenomena (known as chattering). In real applications, it is quite dangerous for the mechanically moving parts along with actuators health, and it also increases the wear tear of the actuators, it may also lead towards total system failure. This is the limitation in the implementation of SMC in sensitive practical systems.

2.7.1 Higher Order Sliding Mode Control (HOSMC)

Regarding the minimization of chattering phenomenon three approaches are listed in the literature. First, the use of saturation control instead of discontinuous one. It ensures the convergence to a boundary of the sliding manifold [26]. However, the precision along with robustness of the sliding mode is partially lost. The second one is the use of observer-based approach [27], it leads toward degradation of robustness. Levant [66] proposed a third approach, high order sliding mode control (HOSMC). It consists derivatives in the sliding variable system. It also maintains the robustness of the system, especially the SOSMC. The drawback present in HOSM is its sensitiveness toward unmodeled fast dynamics. On such scenario, the minute value of chattering may lead towards chaos regarding system stability.

A majority of the non-linear system presents very sensitive response towards very minimal disturbance, even in matched nature, during reaching phase. This response of the system sensitivity during the reaching phase may results in the system instability. Thus the need arises to have a controller which must be free from reaching phase, named as integral sliding mode control [7].

2.7.2 Integral Sliding Mode Control (ISMC)

It is worthy to mention that SMC and HOSM cannot guarantee the invariance property in the necessary reaching phase. Therefore, to mitigate this threat, an integral sliding mode based strategy was proposed in [7] and[28], which established sliding mode without reaching phase. Additionally, it also neglects the uncertainties and disturbances which effects the system and causes unstableness during reaching phase [67]. Another important feature is the same order of motion of equation as the order of the original system, which results robustness of the system is granted throughout an entire response of the system starting from the original time instant [7]. ISMC has a broad scope of applications in robotics, electric drives, electromechanical and underactuated systems. Limited work has done on the ISMC from an underactuated system point of view.

2.8 Research Gap and Challenges Regarding Underactuated Systems

As the importance of underactuated systems are very much evident in the current technological era in light of its diverse health care, industrial and military applications. In this field, new dimensions could be explored in light of [68-71] for set point regulation. Similarly, in tracking control, [63] and [72] can be developed further. Modeling with friction is also remained ignored in the mathematical modeling of under-actuated systems, although it plays a critical role in many precise applications [73-74]. Therefore, aforesaid field is not fully matured yet and still considered to be the most active field of research by the control community.

Since the practical system varies in structure and dynamics, according, the nature of underactuated system varies from system to system. Mostly, approaches listed in literature are system specific approaches which also lacks robustness. SMC is robust, but due to the existence of reaching phase, it is vulnerable towards disturbance rejection. If we increase the order of sliding mode, robustness decreases. Hence, there is a need for improvements in the design to, eliminate reaching phase, ensure robustness from the very beginning, suppress chattering and ensure finite time convergence.

Control algorithms are designed based on two options. One is to adopt the nonlinearities and second is to reduce the system into the lower dynamical model. However, these conditions are not practically possible due to hardware limitations (e.g., actuation power limitation). Challenges are basically divided into two categories, based on their nature, i.e., theoretical and practical challenges. Both categories are discussed in the upcoming section.

2.8.1 Theoretical Challenges

Currently, there are two theoretical challenges faces by control community with respect to underactuated systems. First regarding controllability and stabilization, secondly is related to configuration characteristics of underactuated systems.

In earlier work, it is shown that controllability of system rely on linearization property [75], it means the system can only be controllable if its linearization at an equilibrium point is controllable. Luca et al. [76] provide the solution to the aforesaid constraint by suggesting the time-varying (or discontinuous) feedback controller for stabilization as time-invariant continuous feedback control is not applicable to underactuated systems [77]. This is the reason for discontinuous control like SMC have got great interest by the researchers.

Global stabilization has achieved by Xu et al [63], for the class of underactuated system using discontinues feedback control on systems in cascade form. Olfatai [73] has also done significant work, by proposing the explicit change in coordinates that helps to transform numerous classes of underactuated systems into cascade non-linear systems with structural properties. Still, the window of opportunity is present in the set point regulation problem in light of [16], [39], and [78].

2.8.2 Practical Challenges

Practically underactuated systems remains challenging mainly due to following four aspects,

- regarding industrial needs
- DOF in complex underactuated systems
- fault detection and control
- networked underactuated systems

According to industrial needs, underactuated systems required autonomous operation in an unstructured and possibly dynamic changing environment [79]. As the degree of freedom and complexity increases the reliability of the system decreases. These systems are unable to work in the hazardous or unsafe environment because the cost of damage and loss rate are very high. New tools are required to address robustness issues [80]. Sensor failure may also result as missing feedback signal, affecting the overall tracking problem of the system. Similarly, actuator failure converts the fully-actuated system into the underactuated system. Both problems are well thought by the researchers and recommendation have been made regarding fault detection in such systems [81-82]. If the network is introduced in a loop and if communication signal (data packet) is delayed or network is disconnected.

It may lead towards poor performance of the system, in such case predictive control is required. Hence, there is a gap present with respect to practical aspects of underactuated systems, which needed to be addressed.

2.9 Summary

In this chapter, some historical perspective of different control strategies exercised on underactuated systems are highlighted. These techniques includes, partial feedback linearization, passivity-based control, backstepping, IDA-PBC, fuzzy control, optimal control, and sliding mode control. Most of the aforementioned approaches are system based approaches. It reflects the need to develop the robust control approach for the class of underactuated systems. In light of this chapter, it can be concluded that sliding mode control approach can fulfill the aforesaid requirements to maximum extent. The only serious drawback lies in SMC strategy is the high-frequency switching (known as chattering), due to discontinues function. Complete removal of the chattering phenomenon is not possible; however, it can be minimized. This chattering phenomenon is further analyzed in the upcoming chapters.

In the next chapter, some mathematical preliminaries are presented, which are seems to be necessary for a better understanding of this dissertation.

Chapter 3

Mathematical Preliminaries

If we study the history, there always remains an invisible war between humans and their surrounding (it may be natural or artificial). It is in human's nature to make is surroundings in his favor like, e.g., warm clothes are associated with cold weather, long distance associated with ground or aerial vehicles, self-defense associated with weapon industry. However, this modern era is fenced by a lot of artificial surroundings which includes sophisticated equipment/ devices like, nuclear reactor, humanoids, advanced automotive, etc. As this kind of equipment increases, a natural question regarding their control may arise. This need and kind of control expended too many domains which include, biological, mechanical, chemical, electrical and social control systems. This urge to get control of their surrounding/environment by the mankind give birth to the multidisciplinary subject, today is pronounced as "control systems." A "control systems" is a piece of hardware or software, devise to accomplish a specified task. It can be said that the "control systems" is a framework responsible for deploying available recourses intelligently to meet our requirement/demand. The need for control engineers/researchers are also raised due to imperfections in mathematical models.

From analysis and design point of view, the subject of the control system can categorize into two broad streams, i.e., linear control system and non-linear control system. Linear control system theory deals with linear approximations of nonlinear model and controller has been designed using linear control algorithms in both frequency domain (Bode Plot and Root Locus) and time domain (State Space). Elseways, non-linear control system theory is preferred for the nonlinear models to handle the mathematical imperfections and model uncertainties via nonlinear controllers, e.g., sliding mode control variants, back-stepping and input-output linearization.

The rest of this chapter is organized in such a way that, various terminologies regarding sliding mode control (including examples) are described in Section 3.1. Section 3.2 portrays the introduction of higher order sliding modes (with super-twisting and smooth super-twisting SMC). Section 3.3 displays about the integral sliding mode control. Section 3.4 describes general theory about fast terminal sliding mode control. Section 3.5 summarizes this chapter.

3.1 Sliding Mode Control

S.V. Emelyanov (a Russian theoretician) and his fellow researchers conclude the conventional state feedback methodology lacks in providing effective robustness against nonlinearities. Thus they have developed a variable structure control (VSC) in a mid-nineteenth century [83 - 84]. This variable control structure (so-called sliding mode control) produces very flourishing results in connection with simple state feedback control law. The sliding mode control (see for more detail [7]) is always considered as an effective and efficient approach in control systems because of its invariance in sliding mode, i.e., it results in robustness against uncertainties in sliding mode. The design framework of SMC usually supports r =1 systems.

Conventionally, SMC can be implemented in two phases Reaching Phase and Sliding Phase. The most promising aspect regarding SMC is its imposition towards system states on pre-defined surface also known as a sliding surface. This is called reaching/attraction phase. The manifold is constructed by some hyperplane or by the intersection of the hyperplanes in the state space which are termed as switching surfaces. Due to the discontinuous nature of this controller, it can switch between two different system structures (theoretically with infinite frequency) along this switching surface. In result of this switching, a type of system motion keeps in a place called sliding modes. Ones the system states lie on switching surface, it starts sliding toward the equilibrium, it is remembered as sliding phase [85]. However, this switching causes the well-known chattering phenomenon, which is very dangerous for the actuators of the system. On the other hand, SMC is also sensitive towards the disturbances and the uncertainties in the reaching phase. To overcome these problems, many strategies are proposed, e.g., HOSMC or use of saturation function instead of signum function is suggested for suppression of chattering phenomenon. Similarly, ISMC is proposed to counter the issues during reaching phase. By adoption, any of the aforementioned strategy we have to accept with some compensation, it will be under discussion in the coming units.

Consider a Single Input Single Output (SISO) nonlinear system [86], in state space form presented in equation (3.1).

$$\begin{cases} \dot{x} = f(x,t) + g(x,t)\rho \\ y = h(x,t) \end{cases} \tag{3.1}$$

where $x \epsilon R^n$ representing the states vector, and scalar control input ρ belongs to R. It is also considered smooth.
By assuming, $y^{(i-1)} = \xi_i$ for i = 1, 2, 3 up to relative degree n, the system shown in (3.1) can be transformed in to following system

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \vdots \\ \dot{\xi}_n = \varphi(\xi) + \gamma(\xi)\rho \end{cases} \tag{3.2}$$

For the implementation of SMC two significant steps are needed, first one is the selection/designing of sliding surface and the second one includes the designing of control law. To design a control law, at first step a switching manifold of the following form is considered.

$$\sigma(\xi) = \sum_{i=1}^{n} c_i \xi_i \tag{3.3}$$

Computing the time derivative of (3.3), one may have

$$\dot{\sigma}(\xi) = c_1 \xi_2 + \cdots + c_n(\varphi(\xi) + \gamma(\xi)\rho) \tag{3.4}$$

The control law based on SMC is always composed of two components, i.e., an equivalent control component, and a discontinuous control component. Mathematically, it can be expressed as follows:

$$\rho = \rho_{\text{equivalent}} + \rho_{\text{switching}} \tag{3.5}$$

The reachability condition is satisfied, if the Lyapunov function shown in (3.6), complied the condition laying in (3.7).

$$V = \frac{1}{2}\rho^2(\xi) \tag{3.6}$$

$$\dot{V} = \sigma(\xi)\dot{\sigma}(\xi) \leq 0 \tag{3.7}$$

Aforesaid condition ensures the enforcement of sliding mode asymptotically.

Example

Consider nonlinear model of a simple pendulum [87].

$$ml^2\ddot{\theta} + mgl\sin\theta = \rho \tag{3.8}$$

where m, l, g is the mass, length and force of gravity, respectively, θ represents the pendulum position where ρ is the control input. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$, the state space model of the system can be written as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{g}{l}\sin x_1 + \frac{1}{ml^2}\rho \end{cases} \tag{3.9}$$

In SMC framework step one is to design the sliding surfaces i.e.

$$\sigma = c_1 x_1 + x_2 \tag{3.10}$$

By taking, time derivative of the sliding variable (3.10), we get

$$\dot{\sigma} = c_1 \dot{x}_1 + \dot{x}_2 \tag{3.11}$$

By putting the value of $\dot{x}_1$ and $\dot{x}_2$ from (3.9) to (3.11)

$$\dot{\sigma} = c_1 x_2 - \frac{g}{l}\sin x_1 + \frac{1}{ml^2}\rho \tag{3.12}$$

For equivalent control put $\dot{\sigma} = 0$ in equation (3.12)

$$\rho_{eq} = ml^2\left(-c_1 x_2 + \frac{g}{l}\sin x_1\right) \tag{3.13}$$

and

$$\rho_{sw} = -ml^2 K sign(\sigma) \tag{3.14}$$

Moreover, the sliding mode control can be given as

$$\rho = \rho_{eq} - ml^2 K sign(\sigma) \tag{3.15}$$

In above equation (3.15) K should be greater than the upper bounds of the uncertainties present in the system. For system (3.9) and surface (3.10) reachability condition must be satisfied (existence of sliding mode), if Lyapunov candidate function $V = \frac{1}{2}\sigma^2$, satisfy the condition $\dot{V} = \dot{\sigma}\sigma < 0$. Note that incase of fully-actuated systems control design is relatively easy due to availability of broad range of design techniques as compared to under-actuated systems.

3.2 Higher Order Sliding Mode Control (HOSMC)

HOSMC is the advancement of traditional SMC theory presented by Levant (1993), Emelyanov (1996) and Fridman (1996) respectively [88-90]. If one should keeps the function smooth ($\sigma = 0$), then all of its continuous derivatives of σ in the region of sliding mode is called sliding order 'r'. The r^{th} order sliding mode will be ouccer if a control law with non-linear sliding variable drives not only the sliding variable but also its $r - 1$ consecutive derivatives toward manifold in finite time and stays there eventually, in the presence of bounded disturbance [88], i.e:

$$\sigma = \dot{\sigma} = \ddot{\sigma} \cdots\cdots = \sigma^{r+1} = 0 \tag{3.16}$$

FOSMC is the variable structure system. Therefore, it has some limitations like high-frequency oscillations, which originates chattring effect and the order of relative degree should be one. To overcome this chattering phenomenon different solutions are proposed at different times. One of the solution is the use of saturation function instead of sign function, but it degrades the robustness.

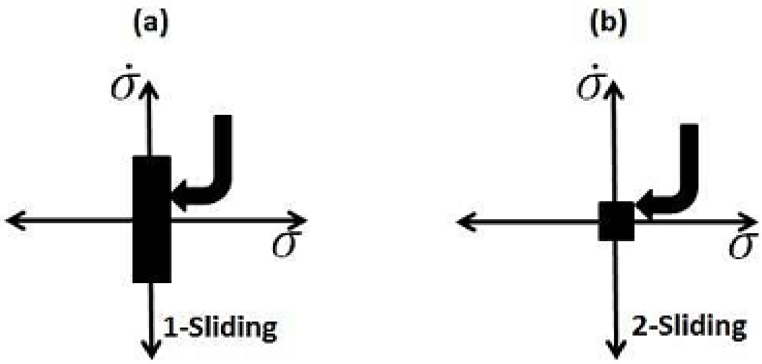

Figure 3.1: Sliding order along with manifold dimension [102].

HOSM can be used to overcome the limitations of standard SMC without losing the robustness. It relaxes the requirement of relative degree and reduces the chattering. This further lead toward not only $\sigma = 0$ but also its time derivative must be zero like $\sigma = \dot{\sigma} = 0$ is known as second order sliding mode control. Increase in sliding order causes decrease in manifold dimensions. This phenomenon can be seen in the Fig. 3.1, first figure on the left represents the conventional or FOSMC while figure on the right displays the SOSM resulting the reduction in the sliding manifold dimensions.

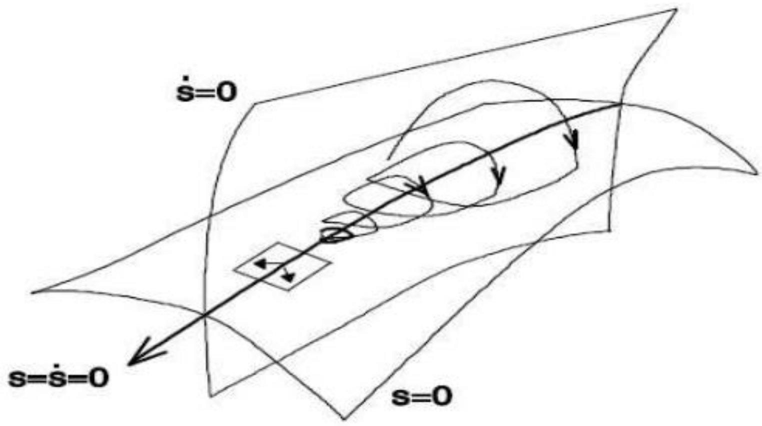

FIGURE 3.2: SOSM Control Trajectory [88].

Second order sliding mode control is the most popular sliding mode control among researchers due to its vast practical applications. The main problem associated with it is the requirement of increased information like availability of σ and $\dot{\sigma}$ (higher derivates). The second order sliding mode trajectory is shown in Fig. 3.2. Other variants of second order sliding mode control includes, Super Twisting Algorithm (STA), Smooth Super Twisting Algorithm (SSTA) and Real Twisting Algorithm (RTA).

3.2.1 Super Twisting Algorithm (STA)

As previously discussed, for SOSM the knowledge of σ and $\dot{\sigma}$ both are requisite. STA is a unique control algorithm in its class which requires only the information of σ [88].

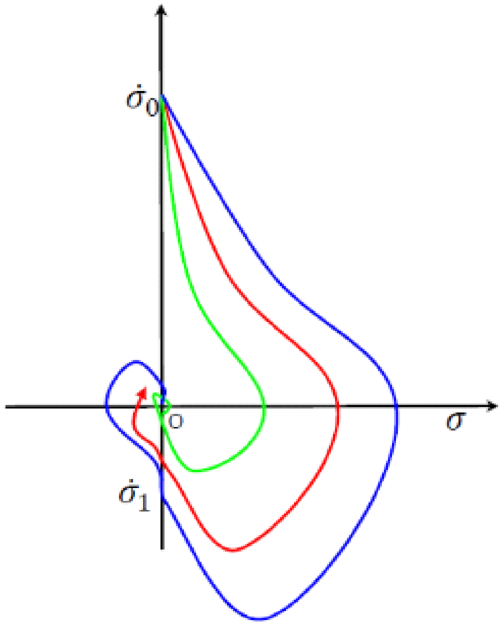

FIGURE 3.3: Phase portrait for Super Twisting Algorithm (STA) [88].

In other words, to divine the STA, derivation of σ is not required. Figure 3.3 shows the phase portrait of STA. The main advantage of STA is that it is SOSM based and non-requirement of sliding surface derivatives. Its practical implementation is comparatively easy. However, one must be clear that the robustness in this technique decreases, as the order of sliding mode increases.

ρ is chosen according to the strategy of [91] as follows:

$$\begin{cases} \rho_i = -k_1 sign(\sigma_i)|\sigma_1|^{\frac{1}{2}} - k_2\sigma_i + \omega_i \\ \dot{\omega}_i = -k_3 sign(\sigma_i) - k_4\sigma_i \end{cases} \tag{3.17}$$

and k_i are positive scalar gains where $i = 1 \cdots$ p is obtained. If k_i are chosen according to [91], Then the enforcement of sliding mode against $\sigma_i = \dot{\sigma}_i = 0$ can be ensured in finite time. However, if $p = 1$ then $k_2 = k_4 = 0$ is usually selected in (3.17), it will appears as follows [91]:

$$\begin{cases} \rho = -k_1 sign(\sigma)|\sigma|^{\frac{1}{2}} + \omega \\ \dot{\omega} = -k_3 sign(\sigma) \end{cases} \tag{3.18}$$

3.2.2 Smooth Super Twisting Algorithm (SSTA)

As much the conventional sliding mode is fascinating for its robustness, meanwhile it is also experience the high-frequency chattering. The beauty of HOSM is the conservation of the features of SMC while reducing the chattering effect. However, HOSM control is sensitive toward unmodeled fast dynamics [92] as a result of which chattering will appear eventually sooner or later in the closed-loop system. The problem associated with the second order sliding mode controllers is the performance degradation, which is coped by the use of smooth SOSMC framework presented in [92-93]. This smooth SMC framework guarantees the effectiveness in many sensitive applications and provides chatter-free smooth control action. The structure of smooth STA (SSTA) is given in the following equation.

$$\begin{cases} \rho = -k_1|\sigma|^{\frac{\mu-1}{\mu}} sign(\sigma) + z \\ \dot{z} = -k_2|\sigma|^{\frac{\mu-2}{\mu}} sign(\sigma) \end{cases} \tag{3.19}$$

Collectively it can be representing as

$$\rho = -k_1|\sigma|^{\frac{\mu-1}{\mu}} sign(\sigma) - \int k_2|\sigma_1|^{\frac{\mu-2}{\mu}} sign(\sigma) \tag{3.20}$$

In above equations (3.19) and (3.20) $\mu \geq 2$. It is worthy to mention here, if $\mu = 2$ the SSTA reduces to conventional STA, where k_1, k_2 are kept strictly positive number.

3.3 Integral Sliding Mode Control (ISMC)

Integral Sliding Mode Control (ISMC) used to avoid chattering and reject uncertainties [85]. The beauty of integral sliding mode control is its freeness from reaching phase which considered to be the necessary part in conventional sliding mode control. It means sliding occurs from the very beginning, which enhances robustness. In integral sliding modes, the system operates with full states while in conventional sliding mode order reduction occur [85]. It also neglects the uncertainties and disturbances, which effects the system and causes instability, during reaching phase. A simple introduction of ISMC is given below.

Consider the nominal system

$$\dot{x} = f(x,t) + g(x,t)\rho \tag{3.21}$$

where $x\epsilon R^n$ is the state vector and $\rho\epsilon R$ represents the control input. It is assumed that the $g(x,\ t)$ is full rank (controllable). In practical applications, if system is not perfectly modeled then system operates with uncertainties caused by variation in parameters and external disturbances. So, equation (3.1) can be reformulate as

$$\dot{x} = f(x,t) + g(x,t)\rho + \xi(x,t) \tag{3.22}$$

where $\xi(x,\ t)$ shows the matched disturbances and it can be represented as

$$\xi(x,t) = g(x,t)\delta \tag{3.23}$$

One thing is evident that the disturbances present in the above nominal system is norm bounded and it would be less than any positive scalar value. For the designing of control law, ρ should be divided in to two parts, ρ_0 and ρ_1, where ρ_0 is being the ideal control and ρ_1 is designed to counter the perturbations. Then (3.24) becomes,

$$\rho = \rho_0 + \rho_1 \tag{3.24}$$

Using equation (3.24) along with (3.22), results (3.25)

$$\dot{x} = f(x,t) + g(x,t)\rho_0 + g(x,t)\rho_1 + \xi(x,t) \tag{3.25}$$

By [8], sliding manifold becomes

$$\sigma(x) = \sigma_0(x) + z \tag{3.26}$$

In equation (3.26), $\sigma_0(x)$ is for the conventional sliding surface, and z represents the integral term. By taking the time derivative of (3.26) along equation (3.25) we got

$$\dot{\sigma} = \nabla\sigma_0[f(x,t) + g(x,t)\rho_0 + g(x,t)\rho_1 + \xi(x,t)] + \dot{z} \tag{3.27}$$

Here we have to choose integral term like the following form

$$\dot{z} = \frac{\partial\sigma_0(x,t)}{\partial x}(f(x,t) + g(x,t)\rho_0) \tag{3.28}$$

$$z(0) = -\sigma_0(x(0)) \tag{3.29}$$

In above equation (3.29), initial condition $z(0)$ is selected to fulfill the requirement of $\sigma(0)$=0. If the condition represented by above equation is satisfies, then sliding mode will occurs from the very beginning. After placing equation (3.28) in equation (3.27), we got

$$\dot{\sigma} = \nabla\sigma_0(g(x,t)\rho_1 + \xi(x,t)) \tag{3.30}$$

To enforce the sliding mode along equation (3.26), the discontinuous control function ρ_1 in equation is selected as

$$\rho_1 = -\left(\nabla\sigma_0\left(g(x,t)\right)\right)^{-1}\left\{M(x)sign(\sigma)\right\} \tag{3.31}$$

Here $M(x)$ can be devised in such a way that its norm should be greater than the norm of uncertainties term and the $det|\nabla\sigma_0 g(x,t)| \neq 0$. The beauty of integral sliding mode control is non-compulsion of reaching phase, enhances robustness. However, chattering further can be reduced using any low pass filter. The design methodology is elaborated with the forthcoming example.

Example

Now again consider the previous example displayed in equation (3.9).

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin x_1 + \frac{1}{ml^2}\rho \end{cases} \tag{3.32}$$

In this example, the control objective is to steer the states to the origin with initial conditions are be set at $x_1(0) = x_2(0) = 0$. According to the procedure described in above section, the control law is composed of two components appear as

$$\rho = \rho_0 + \rho_1 \tag{3.33}$$

In equation (3.33), the second component from the right shows the continuous component, and the first component from the right shows the discontinuous component. The continuous component can be designed by pole placement while assuming that $\sin(x_1) = 0$ i.e., ρ_0 = -$k_1 x_1$-$k_2 x_2$, where k_1, k_2 are the gains, designed using pole placement. However, the discontinuous term can be carried out by assuming an integral manifold of the following form

$$\sigma(x_1, x_2) = c_1 x_1 + x_2 + z = \sigma_0 + z \tag{3.34}$$

By taking derivative (3.34) along (3.9), one has

$$\dot{\sigma} = c_1 x_2 + \left(-\frac{g}{l}\sin(x_1) + \frac{1}{ml^2}(\rho_0 + \rho_1)\right) + \dot{z} \tag{3.35}$$

Choosing $\dot{z} = -\left(c_1 x_2 + \left(-\frac{g}{l}\sin(x_1) + \frac{1}{ml^2}\rho_0\right)\right)$ with $z(0)$=0, the above equation (3.35) becomes

$$\dot{\sigma} = \frac{1}{ml^2}\rho_1 \tag{3.36}$$

By comparing with above equation (3.35) with $\dot{\sigma} = K_1 sign(\sigma)$, one has

$$\rho_1 = -K sign(\sigma) \tag{3.37}$$

where $K = (ml^2)K_1$ is the gain of the discontinuous component. By substituting the continuous and discontinuous part in the expression (3.33), final control law can be achieved. ISMC eliminates the reaching phase and results in the robust regulation of the states to the origin.

3.4 Fast Terminal Sliding Mode Control (FTSMC)

It is evident that the asymptotic convergence in the absence of a strong force may not deliver fast convergence. The conventional terminal sliding mode control, on the other hand, may not confirm fast convergence when the system states have initial conditions entirely away from the equilibrium. However, the fast terminal sliding is capable of combining the advantages of both SMC and Terminal Sliding Mode (TSM) and can make the convergence to the equilibrium faster. Another main aim of the use of this strategy is to acquire high precision tracking with suppressed chattering. The sliding surface of fast terminal sliding mode controller is designed as follows [94]:

$$\sigma(S(\xi)) = \dot{S}(\xi) + \alpha_1 S(\xi) + \beta_1 (S(\xi))^{\frac{p_1}{q_1}} \tag{3.38}$$

where $S(\xi)$ can be defined as:

$$S(\xi) = \sum_{i=1}^{n-1} c_i \xi_i \tag{3.39}$$

The gains α_1 and β_1 in (3.38) are positive constants, p_1 and q_1 are positive odd integers such that q_1 should be greater than p_1. The finite time convergence of FTSMC is very attractive for practical systems due to its high precision results. The main limitation of this strategy towards nonlinear dynamical system is the occurrence of singularity as the order of the system increases.

3.5 Summary

The fundamentals of the theory of SMC, HOSMC, ISMC, and FTSMC are described in this chapter. This primary purpose of this chapter is to provide necessary background which is going to be helpful in studying next chapters.

In next chapter, robust control of the underactuated system is proposed via higher order sliding mode control. Simulation and experimental results reveal the effectiveness of the proposed control strategy.

Chapter 4

Robust Integral Sliding Mode Approach

The control design of underactuated systems was the focus of the researchers in the last decade and also in the existing era. The control design of these kind of systems are quite demanding because of their vital theoretical and practical applications. In addition, another significant feature of underactuated systems is less damaged in case of collision with other objects which in turn provides more safety to actuators. Sliding mode control based global stabilization techniques is proposed by the Xu [63] for the class of underactuated systems in cascaded form, but the problem with sliding mode control is the presence of chattering. The aforesaid design strategy is quite suitable and resulted in satisfactory results, but it is worthy to note that system often becomes too sensitive to disturbance in the reaching phase of sliding mode strategy that the system may even become unstable. Therefore, in order to get rid of this issue the Integral Sliding Mode strategy was proposed [95].

In this chapter, a Robust Integral Sliding Mode Control (RISMC) approach for underactuated systems is proposed by following the footprints of [63]. The benefits of this strategy are the enhancement of robustness from initial time instant. It also suppresses the well-known chattering phenomenon across the manifold. Before the design presentation, the system is suitably transformed into special format. An integral sliding mode strategy is proposed for both the cases along with their comprehensive stability analysis. The proposed technique is practically implemented on a benchmark ball and beam system to validate the effectivity and efficiency of the designed algorithm. Note that, in this chapter my contributions are twofold. The first one is the development of the RISMC and the second one is the practical results of the system on the said system.

This chapter is structured as in Section 4.1 provide the brief description of the problem. In Section 4.2 control law designed is presented. Section 4.3 shows the design procedure implemented on ball and beam (taken as an illustrative example). Section 4.4 and 4.5 displays the simulation and implementation results respectively, which shows the effectiveness of control scheme. At the last conclusion is devised in Section 4.6. In the upcoming section problem statement is presented.

4.1 Problem Statement

Expression presented in [11], for the equation of motion for underactuated systems is expressed below

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = F(q)\rho \tag{4.1}$$

In (4.1), $\mathcal{L}$ represents the represents the systems Lagrangian.Configuration vector is represented by $q \in R^n$, $\rho \in R^m$ denotes the actuator input vector and $F(q) \in R^{n\times m}$ displays the external forces (non-square matrix). In case of $m = rank(F) = n$, (4.1) represents fully actuated system and for $m = rank(F) < n$, system shown in (4.1) acts like an underactuated system. The Lagrangian of system can be expressed as

$$\begin{aligned}\mathcal{L}(q,\dot{q}) &= T(q,\dot{q}) - V(q) \\ &= \frac{1}{2}\dot{q}^{\mathrm{T}}M(q)\dot{q} - V(q)\end{aligned} \tag{4.2}$$

which is the difference between systems kinetic energy $T(q,\dot{q})$ and potential energy. $V(q)$. By [5], the system shown in (4.1) can be written in vector form as following dynamic equation which governs the motion of underactuated system

$$J\left(q\right)\ddot{q}+C\left(q,\ \dot{q}\right)\dot{q}+G\left(q\right)+F\left(\dot{q}\right)=B\left(\rho+\delta\left(q,\ \dot{q},t\right)\right) \tag{4.3}$$

where $q,\dot{q}\in R^{n}$represents the position and velocity states which make a configuration space of $2n$ variable (or states), $J(q)\in R^{n\times n}$ is the inertia matrix, $C\left(q,\ \dot{q}\right)\in R^{n\times n}$ is the matrix describes the centrifugal and coriolis forces, $G\left(q\right)\in R^{n\times 1}$ is gravitational forces and $F(\dot{q})\ \in R^{n\times n}$represents fractional torque. B is the control input channel, and $\rho\in R^{m}$ such that $m<n$ represents the applied control input. The nonlinear term $\left(J^{-1}\ \left(q\right)B\right)$ takes into account the uncertainties in the control input channel. Since we consider the system to be controllable, therefore, before proceeding to the problem formulation we assume that term $\left(J^{-1}\ \left(q\right)B\right)$ is full rank and the origin serves as an equilibrium point. Note that in equation (4.3), $J(q)\ddot{q}$ and $C(q,\dot{q})$ are related as:

$$X=\dot{J}(q)-2C(q,\dot{q}) \tag{4.4}$$

where X represents skew-symmetric matrix, in this regard inertia matric would be $J(q)$ symmetric, therefore we have:

$$\dot{J}(q)=C(q,\dot{q})+C^{T}(q,\dot{q}) \tag{4.5}$$

After partitioning the inertia matrix $J(q)$, the system in (4.3) can be rewritten as follows

$$\begin{cases} j_{11}\ \left(q\right)\ \ddot{q}_1+j_{12}\left(q\right)\ddot{q}_2+h_1\left(q,\dot{q}\right)=0 \\ j_{21}\left(q\right)\ddot{q}_1+j_{22}\left(q\right)\ddot{q}_2+h_2\left(q,\dot{q}\right)=\rho \end{cases} \tag{4.6}$$

where $q=\ \left[q_1\ ,\ q_2\right]^{T}$ represents the states of the system. In order to design a robust control law for the class of underactuated system, the system in (4.3) can be transformed into two formats which are described in the subsequent study.

4.1.1 System in Cascaded Form

Now by following the strategy of [37] and [63], nonlinear system (4.4), can be represented via following cascade form:

$$\begin{cases} \dot{x_1} = x_2 + d_1 \\ \dot{x_2} = f_1\left(x_1,\ x_2\ ,\ x_3\ ,x_4\right) + d_2 \\ \dot{x_3} = x_4 \\ \dot{x_4} = f_2\left(x_1,\ x_2\ ,\ x_3\ ,x_4\right) + b\left(x_1,\ x_2\ ,\ x_3\ ,x_4\right)\rho + d_3 \end{cases} \tag{4.7}$$

where $x_1,\ x_2,\ x_3,\ \ x_4$ are the states of the systems (measurable), x_1 and x_2 are directing towards the position and velocity of the indirect actuated system (4.7) while x_3 and x_4 points towards the position and velocity of the directly actuated system. ρ represents the control input. The nonlinear functions f_1 , f_2: $R^{4n} \rightarrow R^n$, $b : R^{4n} \rightarrow R^{n\times n}$ are smooth in nature. Now, following the procedure of [63], the disturbances d_1, d_2 and d_3 are deliberately introduced to get an approximate controllable canonical form. Note that, practical systems like ball and beam [4], cart-pole system [5], TORA [6], pendubot [39], overhead crane [59] and acrobot [69] can be put in the form presented in (4.7). Before proceeding to the control design of the above cascaded form, the following assumptions are made.

Assumption 4.1

It is assumed that $b\left(x_1,\ x_2\ ,\ x_3\ ,x_4\right)$ is nonzero everywhere in the available space. This assumption confirms the controllability of the given nonlinear system.

Assumption 4.2

Assume that

$$f_1\left(0,0,0,0\right) = 0 \tag{4.8}$$

Equation (4.8) confirms that the origin is an equilibrium point in closed loop.

Assumption 4.3

$\frac{\partial f_1}{\partial x_3}$ is invertible or $\frac{\partial f_1}{\partial x_4}$ is invertible, which, in other words, confirms the controllability of the given nonlinear system.

Assumption 4.4

$f_1(0, 0, x_3, x_4) = 0$is an asymptotically stable manifold, i.e. x_3 and x_4 approaches zero.

Note that the assumption (4.3) and (4.4) lies in the category of non-necessary conditions. These are only used when one needs to furnish the closed-loop system with a sliding mode controller (see for details [63]).

4.1.2 Input Output Form

The system in (4.7) can be transformed into the following input-output form while following the procedure reported in [95]. Let us assume that the system has a nonlinear output $y = h(x)$. To this end we denote

$$\begin{cases} L_f h(x) = \frac{\partial h(x)}{\partial x} f(x) = \nabla h(x) f(x) \\ L_{f_u} h(x) = \frac{\partial h(x)}{\partial x} f_u = \nabla h(x) f_u \end{cases} \tag{4.9}$$

Recursively, it can be written as

$$\begin{cases} L_f^0 h(x) = h(x) \\ L_f^j h(x) = L_f\left(L_f^{j-1} h(x)\right) = \nabla\left(L_f^{j-1} h(x)\right) f(x) \end{cases} \tag{4.10}$$

Assume that the system reported in (4.7) has a relative degree "r" with respect to the defined nonlinear output. Therefore, owing to [96], one has

$$y^{(r)} = {L_f}^r h(x) + L_g({L_f}^{r-1} h(x))\rho + \zeta(x, t) \tag{4.11}$$

subject to the following conditions:

1. $L_g({L_f}^i h(x)) = 0 \ \forall\, x \in B$, where B indicates the neighborhood of x_0 for $i<r-1$;

2. $L_g({L_f}^{r-1}h\,(x)\,)\neq 0$, where $\zeta\,(x,t)$ represents the matched un-modeled uncertainties in (4.11).

System (4.11), by defining the transformation $y^{(i-1)}=\xi_i$ [97], can be put in the following form:

$$\begin{cases} \dot{\xi}_1=\xi_2 \\ \dot{\xi}_2=\xi_3 \\ \vdots \\ \dot{\xi}_n=\varphi(\hat{\xi},\hat{\rho})+\gamma(\hat{\xi})\{\rho^{(k)}+\Delta G_m(\hat{\xi},\hat{\rho},t)\} \end{cases} \tag{4.12}$$

where $k+r=n$, and $\hat{\rho}=(\rho,\dot{\rho},\cdots,\rho^{(k-r)})$, $\hat{\xi}=(\xi_1,\xi_2,\ldots,\xi_n)$ transformed states the so-called phase variables, ρ the control input and $\Delta G_m\left(\hat{\xi},\hat{\rho},t\right)$ represents matched uncertainties. It is worthy to notice that the inverted pendulum and the ball and beam systems can be replaced in the aforementioned form.

Note that, both the formats are ready to design the control law for these systems. In the next section, the procedure for both the forms are outlined.

4.2 Control Law Design

The control design for the forms presented in (4.7) and (4.12), is carried out in this section which we claim as our main contribution in this chapter. The main objective of this work is to enhance the robustness of the system from the very beginning of the process which is the beauty of ISMC. In general, the ISMC law appears as follows [7]. In the subsequent subsections, the author aims to present the design procedure.

4.2.1 Integral Sliding Mode (ISM)

This variant of sliding mode possesses the main features of the sliding mode like robustness and the existence chattering across the switching manifold. On the other hand, the sliding mode occurs from the very start which, consequently, provides insensitivity of disturbance from the beginning. The control law can be expressed as follows. As an extension of traditional sliding mode schemes, the concept of integral sliding mode concentrates on robustness during the entire response. As there is no order reduction takes place, therefore sliding mode is established without reaching phase, implying that the invariance of the system to parametric uncertainty and external disturbances is guaranteed starting from initial time instant. The control law can be expressed as follows

$$\rho = \rho_0 + \rho_1 \tag{4.13}$$

where the first component on the right-hand side of (4.13) governs the systems dynamics in sliding modes whereas the second component compensates the matched disturbances. Now, the aim is to present the design of aforesaid control components.

Control Design for Case-1

This control design for case-1 is the main obstacle in this subsection. To define both the component, the following terms are defined

$$\begin{cases} e_1 = x_1 \\ e_2 = x_2 \\ e_3 = f_1\left(x_1,\ x_2,\ x_3,\ x_4\right) \\ e_4 = \frac{\partial f_1}{\partial x_1}x_2 + \frac{\partial f_1}{\partial x_2}f_1 + \frac{\partial f_1}{\partial x_3}x_4 \end{cases} \tag{4.14}$$

Using these new variables, the components of the controller are designed in the following subsection. For the sake of completeness, the design of this component is worked out via simple pole placement. Following the design procedure of pole placement method, one gets

$$\rho_0 = -k_1e_1 - k_2e_2 - k_3e_3 - k_4e_4 \tag{4.15}$$

where $k_i,\ i = 1,2,3,4$ are the gains of this control component. This control component steers the states of the nominal system to their defined equilibrium. Now, in the subsequent study the design of the uncertainties compensating term is presented. An integral manifold is defined as follows

$$\sigma = c_1e_1 + c_2e_2 + c_3e_3 + e_4 + z = \ \sigma_0 + z \tag{4.16}$$

where $\sigma_0 = c_1e_1 + c_2e_2 + c_3e_3 + e_4$ represents the conventional sliding manifold which is Hurwitz by definition. Now, computing $\dot{\sigma}$ along (4.5), one has

$$\begin{aligned}\dot{\sigma} = {} & c_1(x_2 + d_1) + c_2(f_1 + d_2) + c_3\left(\frac{df_1}{dt}\right) + \frac{d}{dt}\left(\frac{\partial f_1}{\partial x_1}x_2\right) + \frac{\partial f_1}{\partial x_1}\dot{x_2} \\ & + \frac{d}{dt}\left(\frac{\partial f_1}{\partial x_2}f_1\right) + \frac{\partial f_1}{\partial x_2}\dot{f_1} + \frac{d}{dt}\left(\frac{\partial f_1}{\partial x_3}x_4\right) + \frac{\partial f_1}{\partial x_3}f_2 \\ & + \frac{\partial f_1}{\partial x_3}b\rho_0 + \frac{\partial f_1}{\partial x_3}b\rho_1 + \frac{\partial f_1}{\partial x_3}d_3 + \dot{z}\end{aligned} \tag{4.17}$$

Now, choose the dynamics of the integral term as follows:

$$\begin{aligned}\dot{z} = {} & -c_1x_2 - c_2f_1 - c_3\left(\frac{df_1}{dt}\right) - \frac{d}{dt}\left(\frac{\partial f_1}{\partial x_1}x_2\right) - \frac{\partial f_1}{\partial x_1}\dot{x_2} \\ & - \frac{d}{dt}\left(\frac{\partial f_1}{\partial x_2}f_1\right) - \frac{\partial f_1}{\partial x_2}\dot{f_1} - \frac{d}{dt}\left(\frac{\partial f_1}{\partial x_3}x_4\right) - \frac{\partial f_1}{\partial x_3}\rho_0\end{aligned} \tag{4.18}$$

The expression of the term which compensates the uncertainties may be written as follows

$$\rho_1 = -\left(\frac{\partial f_1}{\partial x_3}b\right)^{-1}\left(\frac{\partial f_1}{\partial x_3}f_2 + Ksign\left(\sigma\right)\right) \tag{4.19}$$

The overall controller will look like

$$\rho = -k_1e_1 - k_2e_2 - k_3e_3 - k_4e_4 - \left(\frac{\partial f_1}{\partial x_3}b\right)^{-1}\left(\frac{\partial f_1}{\partial x_3}f_2 + Ksign(\sigma)\right) \tag{4.20}$$

The constants $c_i's$ are control gains which are selected intelligently according to bounds. In the forthcoming paragraph, the stability of the presented integral sliding mode is carried out in the presence of the disturbances and uncertainties. Consider the following Lyapunov candidate function:

$$V = \frac{1}{2}\sigma^2 \tag{4.21}$$

The time derivative of this function along the dynamics (4.14) becomes

$$\begin{aligned}\dot{V} = \sigma\dot{\sigma} = \sigma\Bigg\{ & c_1(x_2 + d_1) + c_2(f_1 + d_2) + c_3\left(\frac{df_1}{dt}\right) + \frac{d}{dt}\left(\frac{\partial f_1}{\partial x_1}x_2\right) + \frac{\partial f_1}{\partial x_1}\dot{x_2} \\ & + \frac{d}{dt}\left(\frac{\partial f_1}{\partial x_2}f_1\right) + \frac{\partial f_1}{\partial x_2}\dot{f_1} + \frac{d}{dt}\left(\frac{\partial f_1}{\partial x_3}x_4\right) + \frac{\partial f_1}{\partial x_3}f_2 \\ & + \frac{\partial f_1}{\partial x_3}b\rho_0 + \frac{\partial f_1}{\partial x_3}b\rho_1 + \frac{\partial f_1}{\partial x_3}d_3 + \dot{z}\Bigg\}\end{aligned} \tag{4.22}$$

The substitution of (4.18 - 4.19) results in the following form

$$\dot{V} \leq -|\sigma|\,\eta_1 < 0 \text{ or } \dot{V} + \sqrt{2}\eta_1\sqrt{V} < 0 \tag{4.23}$$

subject to $K \geq [||\frac{\partial f_1}{\partial x_3}d_3 + c_1 d_1 + c_2 d_2|| + \eta]$.

This expression confirms the enforcement of the sliding mode from the very beginning of the process, i.e., $\sigma \to 0$ in finite time. Now, we proceed to the actual system's stability. If one considers e_1 as the output of the system then, e_2, e_3 and e_4 becomes the successive derivatives of e_1. Whenever, $\sigma = 0$ is achieved, the dynamics of the transformed system (4.14) will converge asymptotically to zero under the action of the control. component (4.15) [4]. That is, in closed loop, the transformed system dynamics will be operated under (4.15) as follows:

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\\ \dot{e}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -k_1 & -k_2 & -k_3 & -k_4\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\end{bmatrix} \tag{4.24}$$

and the disturbances will be compensated via (4.19). The asymptotic convergence of e_1, e_2, e_3 and e_4 to zero means the convergence of the indirectly actuated system (equation 4.7, first two equations) to zero. On the other hand, the states of the directly actuated system (equation 4.7, last two equations) will remain bounded; that is, state of (4.7) will have some nonzero value in order to keep e_1 at zero. Thus, the overall system is stabilized and the desired control objective is achieved.

Control Design for Case-2

The nominal system related to (4.12) can be replaced in the subsequent alternative form

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \vdots \\ \dot{\xi}_r = \chi\left(\hat{\xi}, \hat{\rho}, \rho^{(k)}\right) + \rho^{(k)} \end{cases} \tag{4.25}$$

where $\chi\left(\hat{\xi}, \hat{\rho}, \rho^{(k)}\right) = \varphi\left(\hat{\xi}, \hat{\rho}\right) + \left(\gamma\left(\hat{\xi}\right) - 1\right)\rho^{(k)}$ It is assumed that $\chi\left(\hat{\xi}, \hat{\rho}, \rho^{(k)}\right)$ $= 0$ at $t = 0$ in addition to the next supposition that (4.25) is governed by ρ_0

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \vdots \\ \dot{\xi}_r = \rho_0 \end{cases} \tag{4.26}$$

or

$$\dot{\xi} = A\xi + B\rho_0 \tag{4.27}$$

where

$$A = \begin{bmatrix} 0_{(r-1)\times 1} & I_{(r-1)\times(r-1)} \\ 0_{1\times 1} & 0_{1\times(r-1)} \end{bmatrix}, \; B = \begin{bmatrix} 0_{(r-1)\times 1} \\ 1 \end{bmatrix} \tag{4.28}$$

Once again, following the pole placement procedure, one may have, for the sake of simplicity, the input ρ_0 is designed via pole placement, that is,

$$\rho_0 = -{K_0}^T \xi \tag{4.29}$$

Now to get the desired robust performance, the following sliding manifold of integral type [7] is defined

$$\sigma(\xi) = \sigma_0(\xi) + z \tag{4.30}$$

where $\sigma_0(\xi)$ is the usual sliding surface and z represents the integral term. The time derivative of (4.30) along (4.12) yields

$$\dot{z} = -\left(\sum_{i=1}^{r-1} c_i \xi_{i+1} + \rho_0\right) \tag{4.31}$$

$$z(0) = -\sigma_0(\xi(0)) \tag{4.32}$$

and

$$\rho_1 = \frac{1}{\gamma\left(\hat{\xi}\right)}\left(-\varphi\left(\hat{\xi}, \hat{\rho}\right) - \left(\gamma\left(\hat{\xi}\right) - 1\right)\rho_0 - K sign\sigma\right) \tag{4.33}$$

The sliding mode is being enforced by the control law along the sliding manifold defined in (4.30), where K can be selected in accordance with subsequent stability analysis.

Thus, the final control law becomes

$$\rho = -{K_0}^T \xi + \frac{1}{\gamma\left(\hat{\xi}\right)}\left(-\varphi\left(\hat{\xi}, \hat{\rho}\right) - \left(\gamma\left(\hat{\xi}\right) - 1\right)\rho_0 - K sign\sigma\right) \tag{4.34}$$

Theorem 4.1:

The sliding mode against the switching manifold $\sigma = 0$ can be ensured if the following conditions are satisfied,

$$\begin{cases} |\Delta G_m(y, \rho, t)| \leq \beta_1 \\ K \geq [K_M \beta_1 + \eta_1] \end{cases} \tag{4.35}$$

where η_1 is a positive constant.

Proof:

To prove sliding mode finite time enforcement, differente (4.25) along the dynamics of (4.7), and then substituting (4.34), one has

$$\dot{\sigma}(\xi) = \sum_{i=1}^{r-1} c_i \xi_{i+1} + \rho_0 - K sign\sigma + \gamma\left(\hat{\xi}\right) \Delta G_m \left(\hat{\xi}, \hat{\rho}, t\right) + \dot{z} \tag{4.36}$$

By substituting (4.31) in (4.36) and then rearranging, one obtains

$$\dot{\sigma}(\xi) = -K sign\sigma + \gamma\left(\hat{\xi}\right) \Delta G_m \left(\hat{\xi}, \hat{\rho}, t\right) \tag{4.37}$$

Now, the time derivative of the Lyapunov candidate function $V = \frac{1}{2}\sigma^2$, with the use of the bounds of the uncertainties, becomes

$$\dot{V} \leq -|\sigma| \left[-K + \left|\gamma\left(\hat{\xi}\right) \Delta G_m \left(\hat{\xi}, \hat{\rho}, t\right)\right|\right. \tag{4.38}$$

This expression may also be written as

$$\dot{V} \leq -|\sigma| \eta_1 < 0 \text{ or } \dot{V} + \sqrt{2}\eta_1 \sqrt{V} < 0 \tag{4.39}$$

provided that

$$K \geq [K_M \beta_1 + \eta_1] \tag{4.40}$$

The inequality in (4.39) presents that $\sigma(\xi)$ approaches zero in a finite time t_s [93], such that

$$t_s \leq \sqrt{2}{\eta_1}^{-1}\sqrt{V}\left(\sigma(0)\right) \tag{4.41}$$

which completes the proof.

4.3 Illustrative Example

The control algorithms presented in this section is applied to the control design of a benchmark system (ball and beam). The assessment of the proposed controller,

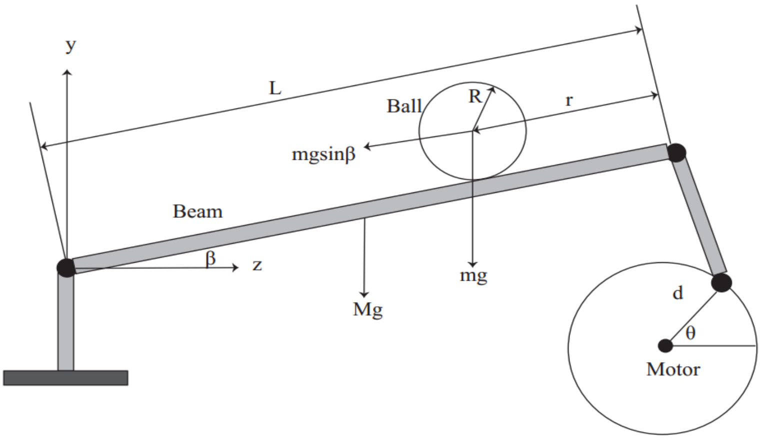

FIGURE 4.1: Schematic Diagram of the Ball and Beam System.

for the ball and beam system, is carried out on the basis of output tracking, robustness enhancement via the elimination of reaching phase and chattering free control input in the presence of uncertainties [95, 98].

4.3.1 Description of the Ball and Beam System

The ball and beam system is a very sound candidate for the class of underactuated nonlinear system. It is famous because of its nonlinear nature and due to its wide range of applications in the current era like passenger cabin balancing in luxury cars, balancing of liquids, balancing of liquid fuel in vertical take-off objects. In term of control scenarios, it is an ill-defined relative degree system which, to some extent, do not support input-output linearization. A schematic diagram with their typical parameters of the ball and beam system is displayed in the adjacent Fig. 4.1 and Table 4.1, respectively.

The equipment used in this study is manufactured by GoogolTech. In general, this system is equipped with a metallic ball, which is let free to roll on a rod having a specified length having one end fixed and the other end is allowed to move up and down via an electric servo motor. The position of the ball can be measured via different techniques. The measured position is feedback to the system and accordingly the motor which works like an actuator which moves the ball up and down and makes balanced the ball at the user defined position. The

TABLE 4.1: Parameters and values used in equations.

Symbol	Quantity	Units	Values
g	Gravitational acceleration	m/s^2	9.81
m	Ball mass	kg	0.04
M	Beam mass	kg	0.15
L	Beam length	m	0.4
R_m	Motor armature resistance	Ω	9
J_m	Motor moment of inertia	Nm/(rad/s^2)	7.35×10^{-4}
C_m	Motor torque constant	Nm/A	0.0075
C_g	Ratio of gear	–	4.28
d	Radius of arm connected to servo motor	m	0.04
J_1	Beam moment of inertia	kgm^2	0.001
C_b	Back *emf* value	V/(rad/s)	0.5625

motion governing equation of this system are given below which are adapted from [4, 28-29].

$$\begin{cases} (mr^2 + T_1)\,\ddot{\beta} + (2mr\dot{r} + \ T_2)\,\dot{\beta} + \left(mgr + \frac{L}{2}Mg\right)\cos\beta \ = \rho \\ T_4\ddot{r} - r\dot{\beta}^2 + gsin\ \beta = 0 \end{cases} \tag{4.42}$$

where $\theta\,(t)$ angle subtended to make stable the ball, the lever angle is represented by $\beta\,(t)$, $r\,(t)$ is the position of the ball on the beam and $v_{in}\,(t)$ is the input voltage of the motor whereas the controlled input appears mathematically via the expression $\rho\,(t) = T_3 v_{in}(t)$ in the dynamic model.

The derived parameters used in the dynamic model of this system are represented by $T_1, T_2,\ T_3$ and T_4 with the following mathematical relations [4].

$$\begin{cases} T_1 = \frac{R_m \times J_m \times L}{C_m \times C_b \times d} + J_1 & T_3 = 1 + \frac{C_m}{R_m} \\ T_2 = \frac{L}{d}\left(\frac{C_m \times C_b}{R_m} + C_b + \frac{R_m \times J_m}{C_m \times C_g}\right) & T_4 = \ \frac{7}{5} \end{cases} \tag{4.43}$$

The equivalent state space model of this is described as follows by assuming $x_1 =$

r (position of ball), $x_2 = \dot{r}$ (rate of change of position), $x_3 = \beta$ (beam angle) and $x_4 = \dot{\beta}$ represents the rate of change of angle of the motor.

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = \frac{1}{T_4}(-gsin(x_3)) \\ \dot{x_3} = x_4 \\ \dot{x_4} = \frac{1}{mx_1^2+T_1}(\rho - (2mx_1x_2 + T_2)\,x_4 - \begin{pmatrix} mgx_1 \\ +\frac{L}{2}Mg \end{pmatrix} cos\ x_3) \end{cases} \tag{4.44}$$

Now, the output of interest is $y = x_1$,which represents the position of the ball. This representation is similar to that reported in (4.7). In the next discussion, the controller design is outlined.

4.3.2 Controller Design

Following, the procedure outlined in Section 4.2,

$$\begin{cases} y = x_1, \\ \dot{y} = x_2, \\ \ddot{y} = -\frac{g}{T_4} sin(x_3), \\ y^{(3)} = -\frac{g}{T_4} x_4 cos(x_3), \\ y^{(4)} = \dfrac{1}{T_4\,(mx_1^2 + T_1)} \Big[-\rho cos\ x_3 + (2mx_1x_2 + T_2)\,x_4 cos\ x_3 \\ \qquad + \left(mgx_1 + \dfrac{L}{2}Mg \right) cos^2 x_3 + {x_4}^2\,(mx_1^2 + T_1)\,\mathrm{sin} x_3 \Big], \end{cases} \tag{4.45}$$

$$y^{(4)} = f_s + h_s\rho, \tag{4.46}$$

where

$$\varphi(\xi) = f_s = \frac{g}{T_4}\left[\frac{(2mx_1x_2 + T_2)\,x_4 + \left(mgx_1 + \frac{L}{2}Mg\right) cosx_3}{mx_1^2 + T_1} \times cosx_3 + x_4^2 sinx_3 \right]$$

and

$$\gamma(\xi) = h_s = \frac{-gcos\ x_3}{T_4(mx_1^2 + T_1)}$$

Now, writing this in the controllable canonical form (phase variable form), one may have

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \vdots \\ \dot{\xi}_4 = \varphi\left(\hat{\xi}\right) + \gamma\left(\hat{\xi}\right)\rho + \gamma\left(\hat{\xi}\right)\Delta G_m\left(\hat{\xi}, \rho, t\right) \end{cases} \tag{4.47}$$

where

$$y^{(i-1)} = \xi_i, \tag{4.48}$$

$\gamma(\hat{\xi})\Delta G_m(\hat{\xi}, \rho, t)$ represents the matched model uncertainties. Moreover, it is observed that the relative degree is 4, therefore, system bears no input derivatives. Here we discuss ISMC on ball and beam system with fixed step tracking as well as variable step tracking. The integral manifold is defined as follows

$$\sigma = c_1\xi_1 + c_2\xi_2 + c_3\xi_3 + \xi_4 + z \tag{4.49}$$

The time derivative of above expression develops into

$$\dot{\sigma} = c_1\dot{\xi}_1 + c_2\dot{\xi}_2 + c_3\dot{\xi}_3 + \varphi(\hat{\xi}) + \gamma(\hat{\xi})\rho_0 + \gamma(\hat{\xi})\rho_1 + \dot{z} \tag{4.50}$$

and $\dot{z}$ appears as

$$\dot{z} = -c_1\xi_2 + c_2\xi_3 + c_3\xi_4 - \varphi\left(\hat{\xi}\right) - \gamma\left(\hat{\xi}\right)\rho_0 \tag{4.51}$$

The expression of the overall controller becomes

$$\rho = -k_1\xi_1 - k_2\xi_2 - k_3\xi_3 - k_4\xi_4 + \frac{1}{\gamma\left(\hat{\xi}\right)}\left\{-\varphi\left(\hat{\xi}\right) - \left(\gamma\left(\hat{\xi}\right) - 1\right)\rho_0 - Ksign\sigma\right\} \tag{4.52}$$

For aiming reference tracking, integral manifold and the controller will appear as follows

$$\sigma = c_1 (\xi_1 - r_d) + c_2\xi_2 + c_3\xi_3 + \xi_4 + z \tag{4.53}$$

$$\rho = -k_1(\xi_1 - r_d) - k_2\xi_2 - k_3\xi_3 - k_4\xi_4 + \frac{1}{\gamma\left(\hat{\xi}\right)}\left\{-\varphi\left(\hat{\xi}\right) - \left(\gamma\left(\hat{\xi}\right) - 1\right)\rho_0 - K sign(\sigma)\right\} \tag{4.54}$$

where r_d is the desired reference with $\dot{r_d}, \ddot{r_d}, \dddot{r_d}$ are bounded.

4.4 Simulation Results

The simulation study of the system is carried by considering the reference tracking of a square wave signal and sinusoidal wave signal. In the subsequent paragraph, their respective results will be demonstrated in detail.

In case the efforts are directed to track a fixed square wave signal in the presence of disturbances, the initial conditions of the system were set to $x_1(0) = 0.4$, $x_2(0) = x_3(0) = x_4(0) = 0$. Furthermore, the square wave was defined in the simulation code as follows

$$r_d(t) = \begin{cases} 20cm & 0 \le t \le 19 \\ 14cm & 20 \le t \le 39 \\ 20cm & 40 \le t \le 60 \end{cases} \tag{4.55}$$

The gains of the proposed controller used in (4.34) are chosen according to the Table 4.2 shown below:

TABLE 4.2: Parametric values used in the square wave tracking.

Constants	C_1	C_2	C_3	K_1	K_2	K_3	K_4	K
Values	1.2	1.2	0.11	402.98	250.18	60	4.1	5

The output tracking performance of the proposed control input, when a square wave is used as desired reference output, is shown Fig. 4.2. It can be clearly examined that the performance is very appealing in this case. The corresponding sliding manifold profile is displayed in Fig. 4.3 which clearly indicates that the

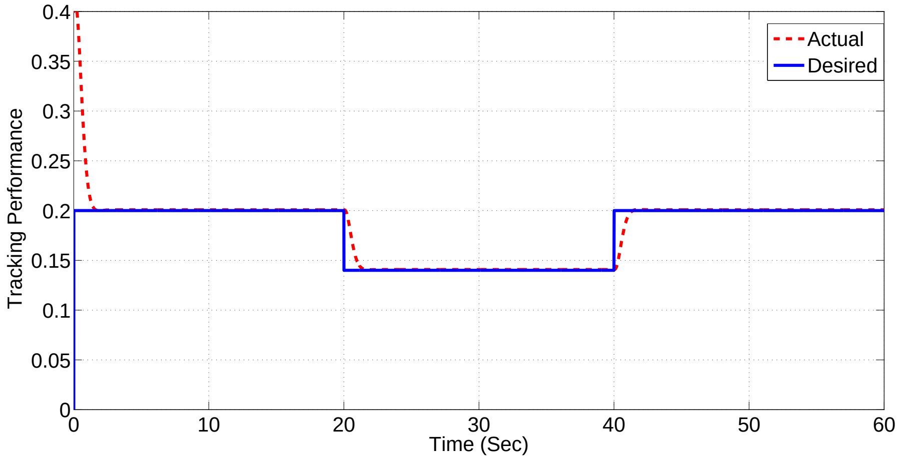

FIGURE 4.2: Output tracking performance when a square wave is used as reference/desired output.

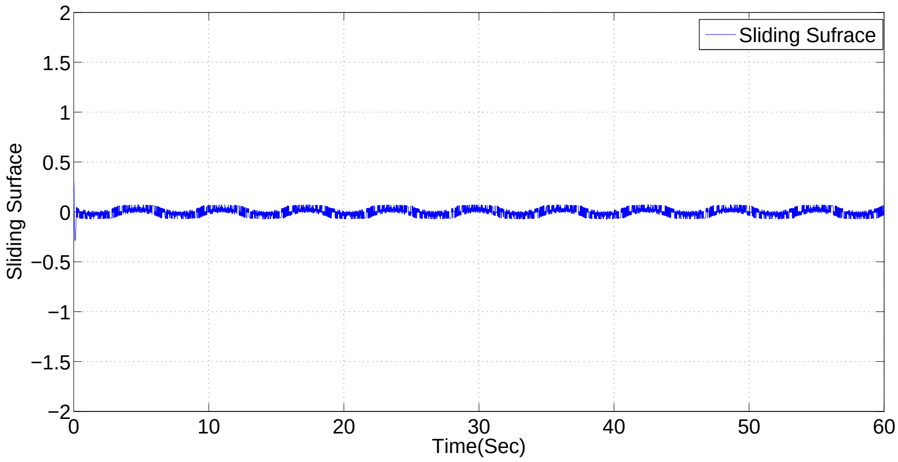

FIGURE 4.3: Sliding manifold convergence profile in case of square wave tracking.

sliding mode is established form the very beginning of the processes which in turn results in enhanced robustness.

The controlled input signal's profiled is depicted in Fig. 4.4 with its zoomed profile is shown in Fig. 4.5. It is evident from both the figures that the control input derives the system with suppressed chattering phenomenon which is tolerable for the system actuators health. Now, from this case study, it is concluded that integral sliding mode approach is an interesting candidate for this class.

In this case study, once again, efforts are focused on the tracking of a sinusoidal signal, which is defined as $r_d(t) = \sin(t)$, in the presence of disturbances. Like

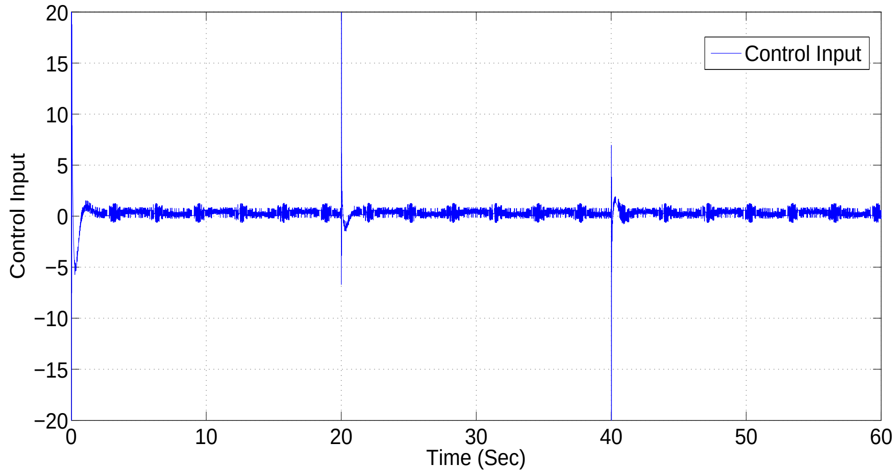

FIGURE 4.4: Control input in square wave reference tracking.

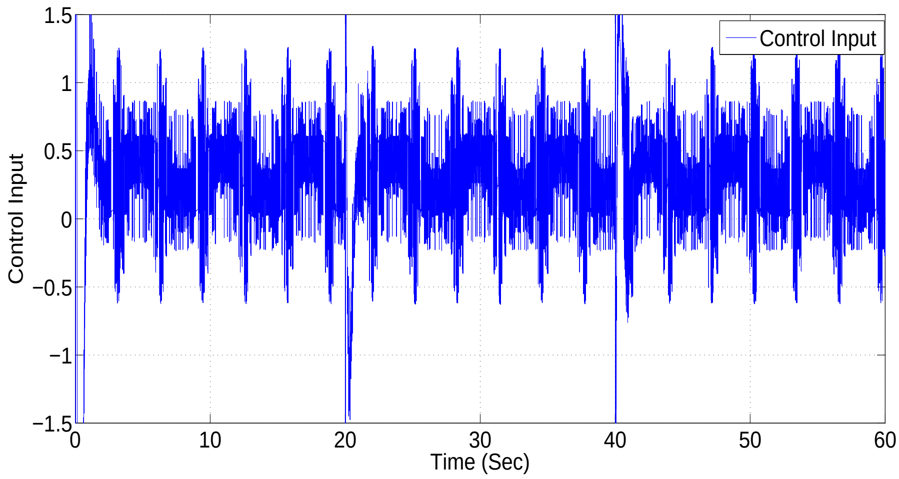

FIGURE 4.5: Zoom profile of the control input depicted in figure 4.4.

the previous case study, the initial condition of the system was set to $x_1(0) = 0.4,\ \ x_2(0) = x_3(0) = x_4(0) = 0$. In addition, the gains of the proposed controller presented in (4.54) are chosen according to the Table 4.3, presented below:

TABLE 4.3: Parametric values used in the sinusoid wave tracking.

Constants	C_1	C_2	C_3	K_1	K_2	K_3	K_4	K
Values	1.2	1.2	0.11	402.98	250.18	230	4.9	5

The output tracking performance of the proposed control input, when a sinusoidal signal is considered as desired reference output, are shown Fig. 4.6. It can be

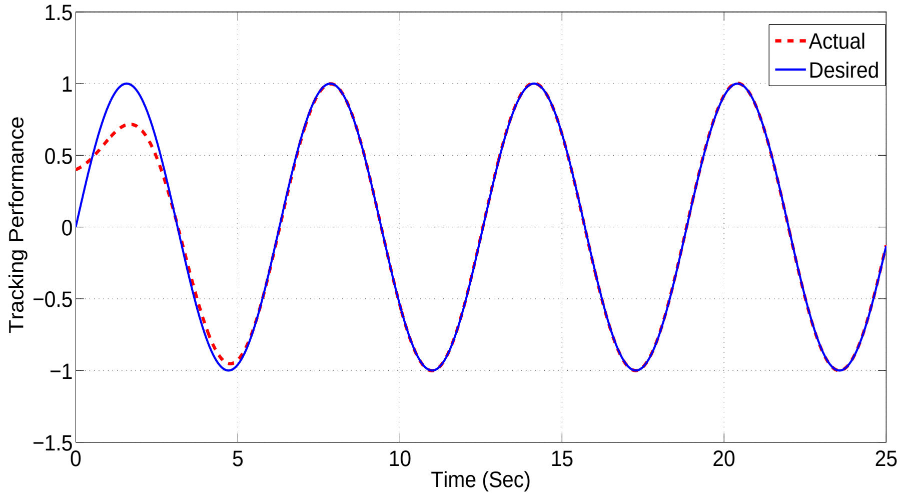

FIGURE 4.6: Output tracking performance when a sinusoidal wave is used as reference output.

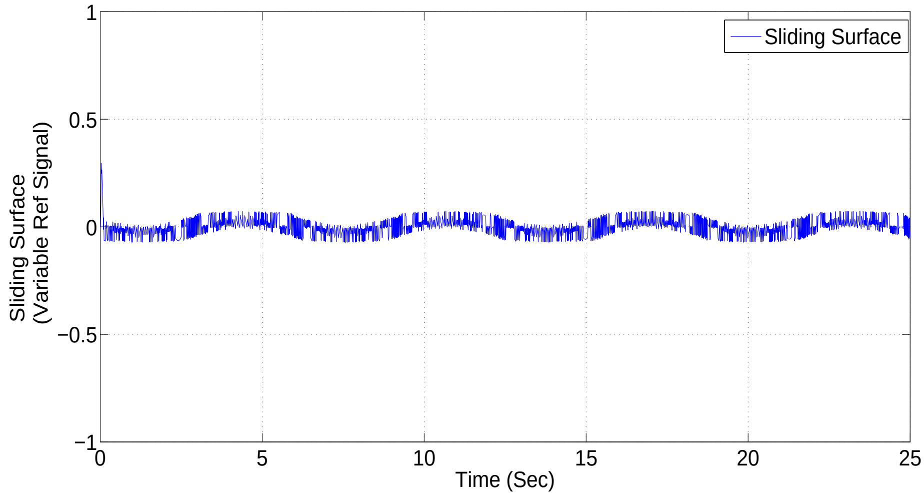

FIGURE 4.7: Sliding manifold convergence profile in case of sinusoidal wave tracking.

clearly seen that the performance is excellent in this scenario. The corresponding sliding manifold profile is displayed in Fig. 4.7 which confirms the establishment of sliding modes from the starting instant and, consequently, enhancement of robustness. The controlled input signal's profile is depicted in Fig. 4.8.

It is evident from the figures that the control input evolves with suppressed chattering phenomenon which, once again, makes this design strategy a good candidate for the class of these underactuated systems.

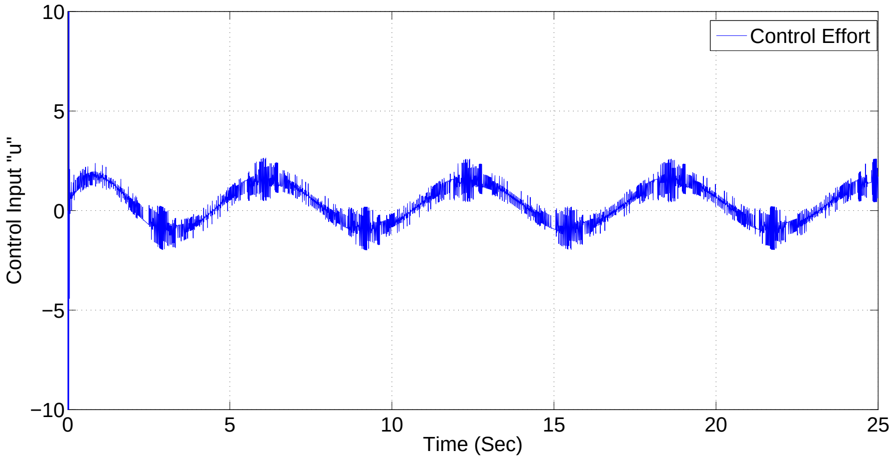

FIGURE 4.8: Control input in sinusoidal wave reference tracking.

4.5 Implementation Results

The control technique proposed in this chapter is implemented on the actual apparatus using the MATLAB environment. The detailed discussions are presented below.

4.5.1 Experimental Setup Description

The experiment setup is equipped by GoogolTech GBB1004 with an electronic control box. The beam length is 40cm along with a mass of the ball, that is, 40g and an intelligent IPM100 servo driver which is used for moving the ball on the beam. The experimental setup is shown in following Fig. 4.9.

The input given to apparatus is the voltage $v_{in}(t)$and the output is the position of the motor $\theta(t)$, which, in other words, is an input for the positioning of the ball on the beam. The motor of the system is capable to rotate clockwise and anticlockwise to stabilize the ball. This apparatus uses potentiometer mounted within a slot inside the beam to sense the position of the ball on the beam. The measured position along the beam is fed to the A/D converter of IPM100 motion drive. The power module used in Googoltech require 220V and 10A input. Note that the control accuracy of this manufactured apparatus lies within the range of $\pm 1mm$.

FIGURE 4.9: Experimental Setup of the Ball and Beam equipped via GoogolTech GBB1004.

The typical parameters values are listed in Table 4.1. The environment used here includes Windows XP as an operating system and MATLAB 7.12/Simulink 7.7. Furthermore, the sampling time used in forthcoming practical results was 2ms. In the experimental processes, the proposed controllers need velocity measurements which are, in general, not available. One may use different kind of velocity observers/differentiator for the velocity estimation. In order to make the implementation easy and simple, a derivative block of the Simulink environment is used to provide the corresponding velocities measurements. In this experiment, the initial conditions were set to $x_1(0) = 0.28, \quad x_2(0) = x_3(0) = x_4(0) = 0$.

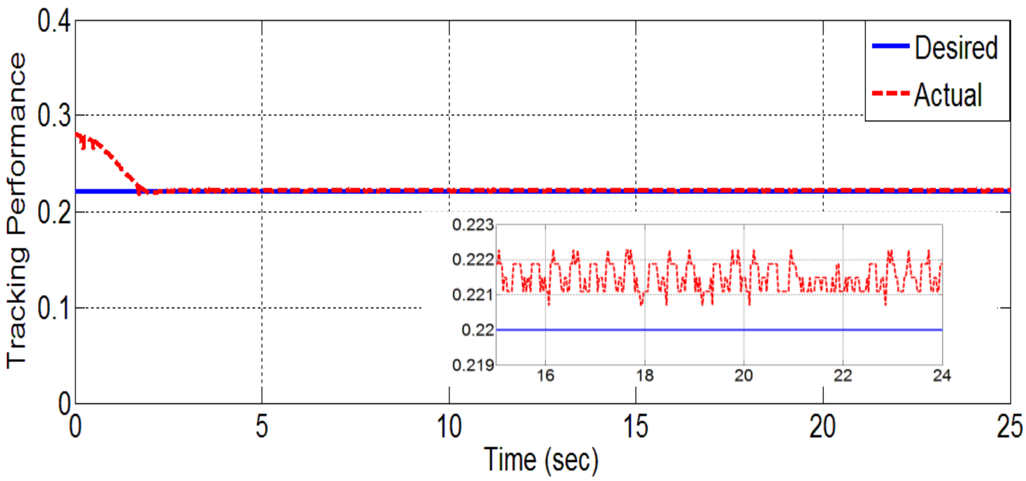

FIGURE 4.10: Output tracking performance when r_d = 22cm is set as reference/desired output.

The reference signal which is needed to be tracked is being defined in (4.53). In Figs. 4.10 and 4.11, the tracking performance is shown. The results reveal that the actual signal $x_1(t)$ is pretty close to the desired signal $r_d(t)$ with a steady state error which is approximately $\pm 0.001m$. The existence of this error is because of the apparatus.

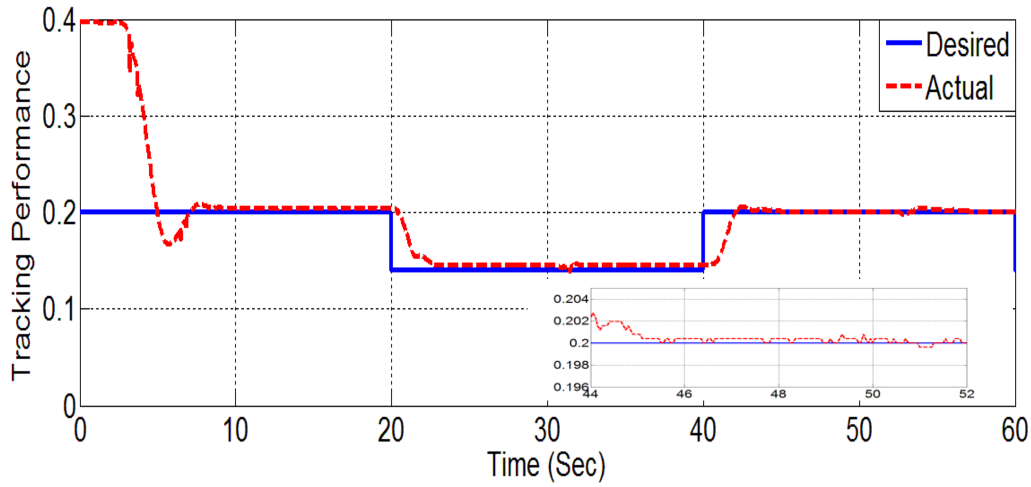

FIGURE 4.11: Output tracking performance when a square wave is used as reference/desired output.

The observations of these tracking results make it clear that the practically implemented results have very close resemblance with the simulation result presented in Fig. 4.2. The error convergence depends on the initial conditions of the ball on the beam. If the ball is placed very close to the desired reference value then it will take little time to reach the desired position. On the other hand, the convergence to the desired will take considerable time if the initial condition is chosen far away from the desired values. This phenomenon of convergence is according the equipment design and structure. The sliding manifold convergence and the control input are shown in Figs. 4.12 and 4.13, respectively. The control input and the sliding manifolds show some deviations in the first second.

This deviation occurs because the ball on the beam, being placed anywhere on the beam, is first moved to one side of the beam and then ball moved to the desired position. The zoomed profile of the control input, being displayed in Fig. 4.14, shows high-frequency vibration (chattering) with the very small magnitude of ± 0.07. This makes the proposed control design algorithm an appealing candidate

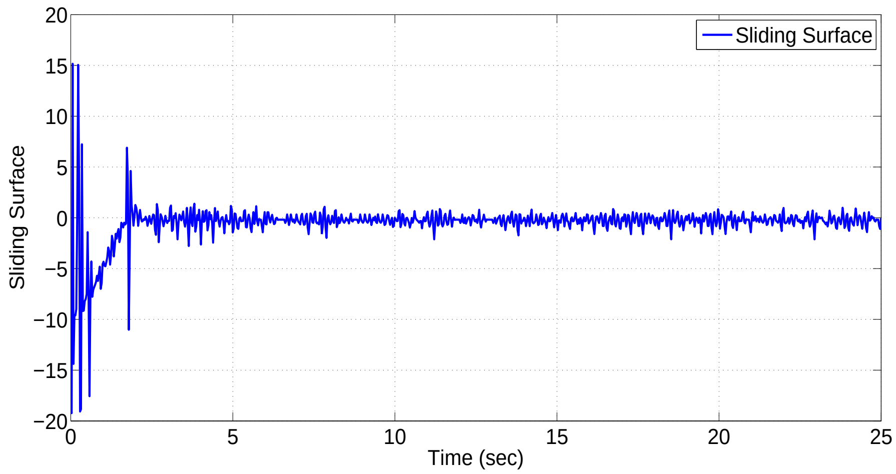

FIGURE 4.12: Sliding surface.

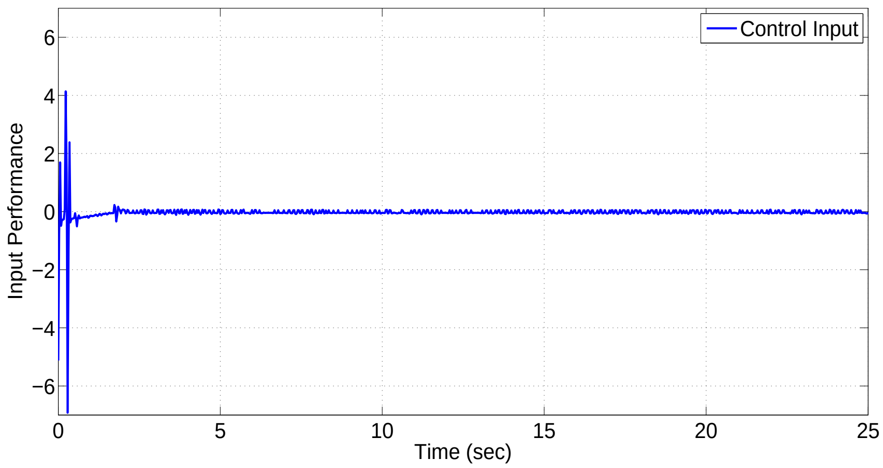

FIGURE 4.13: Control input for reference tracking.

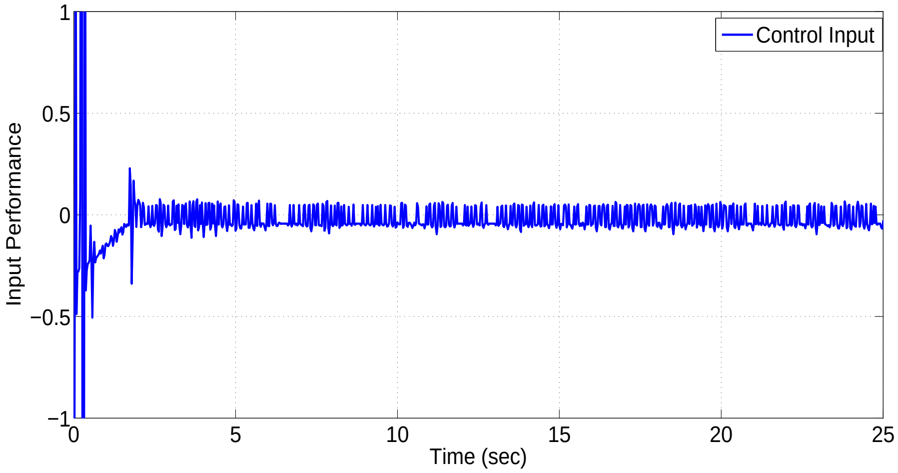

FIGURE 4.14: Zoom profile of the control input depicted in Fig. 4.13.

for this class of nonlinear systems. The gains of the controller being used in this experiment are displayed in Table 4.4.

TABLE 4.4: Parametric values used in implementation.

Constants	C_1	C_2	C_3	K_1	K_2	K_3	K_4	K
Values	8	5	1	3	15	3	1	4

4.6 Summary

The control of underactuated systems, because of its fewer number of actuators than the degree of freedom, is an interesting objective among the researchers. In this chapter, an integral sliding mode control approach, due to its robustness from the very start of the process, is employed to the control design of this class. The design of the integral manifold relied upon a transformed form. The benefit of the transformed form is that it makes easy and simple the design strategy along with its applicability towards the class of underactuated systems which includes,the systems like ball and beam, TORA, inertial wheel pendulum, acrobot, overhead crane, cart-pole system and pendubot.The stability analysis and experimental results of the proposed control laws are presented, which convey the good features and demand the proposed approach when the system operates under uncertainties.

In the upcoming chapter, aforesaid control technique is compared with other SMC variants, and the RISMC shows the promising results among them.

Chapter 5

A Comparative Experimental Study of Robust Sliding Mode Control Strategies for Underactuated Systems

This chapter presents a comprehensive comparative study for the tracking control of a class of underactuated nonlinear uncertain systems. A given nonlinear model of the underactuated system is, at first stage, transformed into an input-output form and the driving applied control input of the transformed system is then designed via four sliding mode control strategies, i.e., conventional first-order sliding mode control, SOSMC, FTSMC, and ISMC. At the second stage, a ball and beam system is considered and the aforementioned four control design strategies are experimentally implemented. A comprehensive comparative study of the simulation and experimental results is then conducted, which take into account the tracking performance [95, 98], i.e., settling time, overshoots, robustness enhancement, chattering reduction, sliding mode convergences, and control efforts.

It is worthy to mention here that in this chapter contributions are in three-fold, i.e., the system transformation into canonical form by defining a suitable output, the simulation as well as practical implementation of the four control strategies and the comparative analysis of the said techniques. The rest of the chapter is presented in a manner like, section 5.1 portrays the problem formulation, where control law designs of the SMC, SOSMC, ISMC, and FTSMC are presented in Section 5.2. The aforesaid control strategies are being exercised on the illustrative example in Section 5.3, where simulation are displayed in Section 5.4. Section 5.5 portrays the experimental results,where conclusion is derived in Section 5.6.

5.1 Problem Formulation

The dynamic equation which governs the motion of underactuated system as discussed in chapter 4 is represented as:

$$J(q)\ddot{q} + C(q,\ \dot{q})\dot{q} + G(q) + F(\dot{q}) = B(\rho + \delta(q,\ \dot{q}, t)) \tag{5.1}$$

Moreover, the system (5.1) in cascaded form can be written as follows [24]:

$$\begin{cases} \dot{x_1} = x_2 + d_1 \\ \dot{x_2} = f_1(x_1,\ x_2\ ,\ x_3\ , x_4) + d_2 \\ \dot{x_3} = x_4 \\ \dot{x_4} = f_2(x_1,\ x_2\ ,\ x_3\ , x_4) + b(x_1,\ x_2\ ,\ x_3\ , x_4)\rho + d_3 \end{cases} \tag{5.2}$$

where $x_1,\ x_2,\ x_3,\ \ x_4$ are the available states of the systems, d_1, d_2 and d_3 are bounded disturbances, $f_1(\ x_1,\ x_2,\ x_3,\ \ x_4)$ and $f_2(\ x_1,\ x_2,\ x_3,\ \ x_4)$ are nonlinear smooth functions. The nonlinear smooth function $b(\ x_1,\ x_2,\ x_3,\ \ x_4)$ represents the control input channel and ρ is the applied controlled input.The description and assumptions for system (5.1) and (5.2) are same as discussed in chapter 4.

Remark 5.1

Since every control methodology can be easily employed to controllable canonical forms, therefore, we aim to transform the newly established form (5.2) to a controllable canonical form. In this way, one may eliminate the required condition on the disturbances d_1, d_2 and d_3 [63]. In addition, one can easily employ the so far available techniques. Such formats are applicable for nonlinear systems like ball and beam [4], cart-pole system [5], TORA [6], pendubot [39], overhead crane [59] and acrobot [69].

Now, by following the procedure defined in section 4.1, the system shown in (5.2) can be transformed into the following input-output form:

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \vdots \\ \dot{\xi}_n = \varphi\left(\hat{\xi}, \hat{\rho}\right) + \gamma\left(\hat{\xi}\right)\{\rho^{(k)} + \Delta G_m\left(\hat{\xi}, \hat{\rho}, t\right)\} \end{cases} \tag{5.3}$$

where $k + r = n$, and $\hat{\rho} = \left(\rho, \dot{\rho}, \cdots, \rho^{(k-r)}\right)$. In (5.3), state vectors and matched uncertainties are represented by $\hat{\xi} = [\xi_1, \xi_2, \ldots, \xi_n]$ and $\Delta G_m\left(\hat{\xi}, \hat{\rho}, t\right)$, respectively. The symbol ρ is the applied control input.

Note that the nonlinear dynamics of an inverted pendulum, double inverted pendulum, ball and beam system, and flexible joints manipulator of link 1 can be easily replaced in the above equivalent input-output form. Before the design it is suitable to assume that:

Assumption 5.1

$$\left|\Delta G_m\ (\hat{\xi}, \hat{\rho}, t)\right| \leq \mathit{\Gamma} \tag{5.4}$$

In a realistic sense, this assumption means that the uncertainty has a tolerable magnitude. Now, the problem in hand is the design of a controller for system (5.3). Having controlled (5.3) will imply a clear solution to the control problem of system (5.2). The core control problem of system (5.3) is to steer the real state to zero, i.e., a regulation problem is considered. This task is fulfilled via a family of control strategies, i.e., FOSMC, SOSMC, ISMC, and FTSMC while considering the system subject to matched disturbances. At this stage, we are now ready to pursue its control design via the aforementioned family of sliding mode controllers.

5.2 Control Law Design

In this section, the control design for the system (5.3) is presented via a family of sliding mode control strategies. Here we proceed by designing the control law via FOSMC, SOSMC, ISMC, and FTSMC.

5.2.1 Sliding Mode Control

The sliding mode control (see for more detail [7]) is always considered as an effective and efficient approach in control systems because of its invariance in sliding mode, i.e., it results in robustness against uncertainties in sliding mode. The design of SMC usually supports systems which have relative degree one with respect to the sliding manifold. The control law is always composed of two components, i.e., an equivalent control component, and a discontinuous control component. Mathematically, it can be expressed as follows:

$$\rho = \rho_{eq} + \rho_{sw} \tag{5.5}$$

In order to design a control law, at first step, a switching manifold of the following form is considered.

$$\sigma(\xi) = \sum_{i=1}^{n} c_i \xi_i \tag{5.6}$$

Computing the time derivative of (5.6) along (5.3) one may have

$$\dot{\sigma}\left(\xi\right) = c_1 \xi_2 + ... + c_i \xi_{i+1} + \varphi\left(\hat{\xi}, \hat{\rho}\right) + \gamma\left(\hat{\xi}\right)\rho \tag{5.7}$$

Now by posing $\sigma(\xi) = 0$, one gets

$$\rho_{eq} = -\frac{1}{\gamma\left(\hat{\xi}\right)}\left(\varphi\left(\hat{\xi},\hat{\rho}\right) + \sum_{i=1}^{n} c_i\xi_{i+1}\right) \tag{5.8}$$

To design the discontinuous control component, a Lyapunov of the following form is defined.

$$V(\xi) = \frac{1}{2}\sigma^2 \tag{5.9}$$

Calculating the time derivative of this function along (5.3) and then substituting (5.8), one may get

$$\dot{V}(\xi) = \sigma(\xi)\left(\gamma\left(\hat{\xi}\right)\Delta G_m(\hat{\xi},\hat{\rho},t) + \rho_{sw}\right) \tag{5.10}$$

Now by choosing the expression of the discontinuous term as follows:

$$\rho_{sw} = -K sign(\sigma) \tag{5.11}$$

By using the bound of uncertainties (5.4), one has

$$\dot{V}(\xi) \leq -|\sigma|\left[-K + \left|\gamma\left(\hat{\xi}\right)\Delta G_m(\hat{\xi},\hat{\rho},t)\right|\right] \tag{5.12}$$

This can also be expressed as

$$\dot{V}(\xi) \leq -|\sigma|\,\vartheta \leq 0 \tag{5.13}$$

provided that

$$K \geq [K_m \mathit{\Gamma} + \vartheta] \tag{5.14}$$

whereϑ and $\mathit{\Gamma}$ are positive constants and K_m is the maximum absolute value of $\gamma\left(\hat{\xi}\right)$. The inequality in (5.13) confirms that $\sigma(\xi)$ approaches zero in a finite time t_s [87]. Consequently, the states of the system (5.3) will be steered to the origin via the control law defined in (5.5) with detailed expressions in (5.8) and (5.11). In SMC, the controller suffers from high frequency vibration in sliding mode phase. In order to reduce this dangerous vibration, in the next subsection, the same problem is handled with second order sliding modes.

Remark 5.2

The most prominent advantage of the first order SMC is the order reduction in sliding mode. This order reduction results in insensitivity to disturbances and model uncertainties. However, to keep the sliding mode, the control input has to switch with infinite frequency along a sliding constraint. This switching causes severe damage to the system components. This characteristic is no more advantageous in the real word and even degrades the sliding modes fascination.

5.2.2 Second Order Sliding Mode Control

Since the drawback/limitation of the conventional SMC is chattering effect [99], which degrades the performance of the system and may lead towards system instability. Therefore, the chattering suppression/removel was focused by a vast number of researchers. In literature, the saturation function is used instead of the discontinuous function [26]. However, in this case, the robustness, as well as accuracy, is partially lost. The other mainstream approach was the use of an observer-based approach which results in less magnitude of the uncertain term [27]. Consequently, the chattering was suppressed.

In the context of chattering removal, the most famous approach was the HOSM control technique [104]. In this approach, the sliding mode occurs along the intersection of the sliding variable and its derivative of order r. In this case, the sliding set is defined to be $\sigma = \dot{\sigma} = \ddot{\sigma} = \dddot{\sigma} = \cdots = \sigma^{(r-1)} = 0$. The structure of the controller is designed in such a way that it confirms finite time enforcement of sliding mode along the defined sliding set in the presence of the disturbances/uncertainties which in turn results in increased accuracy of the sliding modes and output convergence. Moreover, the increase in the order of sliding mode results in reduced chattering. However, the robustness decreases. In the literature, the most famous and appealing relative degree one higher order sliding mode controller is the super twisting (STW) which has considerable robustness with acceptable chattering reduction. We now intend to design super-twisting controller for this class of nonlinear systems.

5.2.2.1 Super-twisting Sliding Mode Control

In this design, the sliding set consists of the intersection of hyperplanes $\sigma(\xi) = 0$ and $\dot{\sigma}(\xi) = 0$ i.e., the sliding mode occurs on the following set

$$\sigma(\xi) = \dot{\sigma}(\xi) = 0 \tag{5.15}$$

Since STW deals with relative degree one case [91], therefore, the sliding variable (5.6) is quite suitable for this design strategy. Now, by taking the time derivative of (5.6), along (5.3) one gets

$$\dot{\sigma}(\xi) = \sum_{i=1}^{n-1} c_i \xi_{i+1} \tag{5.16}$$

or

$$\dot{\sigma}(\xi) = c_1\xi_2 + ... + c_i\xi_{i+1} + \varphi\left(\hat{\xi}, \hat{\rho}\right) + \gamma\left(\hat{\xi}\right)\rho \tag{5.17}$$

This can also be realized as

$$\dot{\sigma}(\xi) = \Psi_1(\xi) + \gamma(\hat{\xi})\rho \tag{5.18}$$

where

$$\Psi_1(\xi) = c_1\xi_2 + ... + c_i\xi_{i+1} + \varphi\left(\hat{\xi}, \hat{\rho}\right) \tag{5.19}$$

By following [1], the control law can be expressed as $\rho = \frac{\rho_1 - \Psi_1(\xi)}{\gamma(\hat{\xi})}$, where ρ is chosen according to the strategy of [91] as follows:

$$\begin{cases} \rho_1 = -k_1 sign(\sigma)|\sigma|^{\frac{1}{2}} - k_2\sigma + \omega \\ \dot{\omega} = -k_3 sign(\sigma) - k_4\sigma \end{cases} \tag{5.20}$$

In expression [91], $k_i; i = 1, 2, 3, 4$ are positive gains. If one chose k_i according to [100], then the enforcement of sliding mode against $\sigma(\xi) = \dot{\sigma}(\xi) = 0$ can be ensured in finite time. Moreover, by selecting ($k_2 = k_4 = 0$), equation (5.20) can be reduced further like shown in equation (5.21),[91].

$$\begin{cases} \rho_1 = -k_1 sign(\sigma)|\sigma|^{\frac{1}{2}} + \omega \\ \dot{\omega} = -k_3 sign(\sigma) \end{cases} \tag{5.21}$$

The chattering attenuation is a considerable advantage of the STW and it remains insensitive to bounded perturbations, but these perturbations cannot increase faster than a linear function of time or it can be said that they do not need to be bounded [101]. For the stability and detailed proof, one may read [91]. The sliding mode control strategy remains very sensitive to disturbances in the reaching phase which may decrease the applicability of this technique. Therefore, a reaching phase free sliding mode control was proposed which enhances the robustness from the very beginning and considerably reduces chattering (see for details [7]). In the subsequent study, an integral sliding mode for the system (5.3) is designed.

Remark 5.3

Since the conventional SMC causes wear tear on the system components, therefore, one of the main challenge which was solved via the second order sliding mode was the chattering attenuation. This technique, on one hand, keep the main characteristics of first order SMC, i.e., order reduction and on the other hand, suppresses chattering. In addition, this technique makes easy the practical implementation. However, one must be clear that the robustness of this technique decreases, as the order of sliding mode increase.

5.2.3 Integral Sliding Mode Control

This technique retains the main features of the sliding mode with enhanced robustness against matched disturbances with attenuated chattering across the switching manifold. Generally, the control law for ISM can be expressed as follows:

$$\rho \;=\; \rho_0 \;+\; \rho_1 \tag{5.22}$$

where the first component on the right-hand side of the above equation governs the system dynamics during sliding modes whereas the matched disturbances have been compensated by the second component. The sliding surface of ISM is defined as:

$$\sigma\left(\xi\right) = \sum_{i=1}^{n} c_i \xi_i + z \tag{5.23}$$

Now, by adapting the strategy proposed in [28] (also in previous chapter), the control structure can be chosen as follows:

$$\rho = -{K_0}^T \xi - \frac{1}{\gamma\left(\hat{\xi}\right)} (\varphi\left(\hat{\xi}, \hat{\rho}\right) + \left(\gamma\left(\hat{\xi}\right) - 1\right) \rho_0 + K sign \sigma) \tag{5.24}$$

where

$$\dot{z} = -\left(\sum_{i=1}^{n-1} c_i \xi_{i+1} + \rho_0\right) \tag{5.25}$$

$$\rho_0 = -{K_0}^T \xi \tag{5.26}$$

and

$$\rho_1 = \frac{1}{\gamma\left(\hat{\xi}\right)} (-\varphi\left(\hat{\xi}, \hat{\rho}\right) - \left(\gamma\left(\hat{\xi}\right) - 1\right) \rho_0 - K sign \sigma) \tag{5.27}$$

The initial condition of equation (5.25) is chosen such that the manifold remains at zero at the initial time $t = 0$ i.e., $z\left(0\right) = -\sigma_0(\xi(0))$ should be justified. (detailed is given in previous chapter, also in [28]). This control law (5.24) establishes sliding mode from the very start of the process and confirms the regulation of the states of the system (5.3) to zero under the action of (5.26).

Remark 5.4

This design technique offers a number of advantages. The most promising one is the establishment of sliding mode from the very start of the processes, i.e., no reaching phase happens. Hence the system becomes more robust from the initial time instant. In addition, the unwanted chattering phenomena can be suppressed up to considerable order. However, no order reduction happens which consequently makes the system sensitive to parametric variations. This sensitivity to the parameter can be reduced by designing a continuous control component of the controlled input more cleverly.

5.2.4 Fast Terminal Sliding Mode Control

It is evident that the asymptotic convergence in the absence of a strong force may not deliver fast convergence. The conventional terminal sliding mode control, on the other hand, may not confirm fast convergence when the system states have initial conditions quite away from the equilibrium. However, the fast terminal sliding is capable of combining the advantages of both SMC and TSM and can make the convergence to the equilibrium faster. This job can be done via changing the definition of the switching manifold. Therefore, in this section, the fast terminal sliding mode for the class of underactuated system (5.3) is designed. Another main aim of the use of this strategy is to acquire high precision tracking with suppressed chattering. The sliding surface of fast terminal sliding mode controller is designed as follows [95]:

$$\sigma(S(\xi)) = \dot{S}(\xi)+\alpha_1 S(\xi)+\beta_1(S(\xi))^{\frac{p_1}{q_1}} \tag{5.28}$$

where $S(\xi)$ can be defined as:

$$S\left(\xi\right) = \sum_{i=1}^{n-1} c_i \xi_i \tag{5.29}$$

The gains α_1 and β_1 in (5.28) are positive constants, p_1 and q_1 are positive odd integers such that q_1 should be greater than p_1. The time derivative of (5.28) along (5.3) takes the form:

$$\dot{\sigma}(S(\xi)) = \left(\sum_{i=1}^{n-1} c_i \xi_{i+1} + \varphi\left(\hat{\xi},\hat{\rho}\right) + \gamma\left(\hat{\xi}\right)\rho + \alpha_1 \dot{S}\left(\xi\right) + \frac{p_1}{q_1}\beta_1 \left(S(\xi)\right)^{\frac{p_1}{q_1}-1}\dot{S}\left(\xi\right)\right) \tag{5.30}$$

By following the design strategies [5] and [94], the final control law can be expressed as follows:

$$\rho = -(\ \gamma(\hat{\xi}))^{-1} \left(\sum_{i=1}^{n-1} c_i \xi_{i+1} + \varphi\left(\hat{\xi}, \hat{\rho}\right) + \alpha_1 \dot{S}(\xi) + \frac{p_1}{q_1} \beta_{\ 1} (S(\xi))^{\frac{p_1}{q_1}-1} \dot{S}(\xi) + K sign(\sigma) \right) \tag{5.31}$$

In order to prove the stability of FTSMC for the system (5.3), the readers may follow strategies of [5] and [94]. This design strategy confirms fast convergence of the system's states to the equilibrium with high precision and suppresses chattering phenomena. Now, at this stage, the aforesaid design techniques are required to be tested on an experimental setup. Therefore, in the forthcoming section, all the controllers are implemented on an actual underactuated ball and beam system.

Remark 5.5

The FTSMC use a sliding manifold which not only results in the finite time enforcement of sliding mode, but also confirms the systems states convergence in finite time. This finite time convergence results in high precision which makes this system more appealing in practical systems. Like conventional SMC, its preserve robustness in a sliding mode with considerably reduced chattering. The main disadvantages of this technique is the singularity occurrence as the order of the system increases. In addition, which treating different systems which deal hydrodynamics force result in an unknown sign of the Lyapunov derivative. Hence, stability becomes questionable.

5.3 Illustrative Example

Once again ball and beam system is considered as an illustrative example of the class of underactuated nonlinear systems which are influenced via the control law design in the previous section. A comprehensive comparative simulation and experimental study is the core of the following study.

5.3.1 Description of the Ball and Beam System

Ball and beam system is briefly discussed in the previous chapter in Section 4.3.1. A schematic diagram of benchmark system (ball and beam system) are shown in Fig. 4.1 whereas its typical parameters are listed in Table 4.1, respectively. The motion governing equation of this system is same as shown in the previous chapter as equation (4.41) adopted from [4, 28-29]:

$$\begin{cases} (mr^2 + T_1)\ddot{\beta} + (2mr\dot{r} + \ T_2)\dot{\beta} + \left(mgr + \frac{L}{2}Mg\right)\cos\beta \ = \rho \\ T_4\ddot{r} - r\dot{\beta}^2 + gsin\ \beta = 0 \end{cases} \tag{5.32}$$

The equivalent state space model of (5.32) can be represented as (4.43), given below:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = \frac{1}{T_4}(-gsin(x_3)) \\ \dot{x_3} = x_4 \\ \dot{x_4} = \frac{1}{mx_1^2+T_1}(\rho - (2mx_1x_2 + T_2)\,x_4 - \left(mgx_1 + \frac{L}{2}Mg\right)cos\ x_3) \end{cases} \tag{5.33}$$

5.3.2 Controller Design

Following, the procedure outlined in Section 4.2, Chapter 4.

$$\begin{cases} y = x_1, \\ \dot{y} = x_2, \\ \ddot{y} = -\frac{g}{T_4}sin(x_3), \\ y^{(3)} = -\frac{g}{T_4}x_4cos(x_3), \\ y^{(4)} = \dfrac{1}{T_4\,(mx_1^2 + T_1)}\Big[- \rho cos\ x_3 + (2mx_1x_2 + T_2)\,x_4cos\ x_3 \\ \qquad + \left(mgx_1 + \dfrac{L}{2}Mg\right)cos^2x_3 + {x_4}^2\left(mx_1^2 + T_1\right)\sin x_3 \Big], \end{cases} \tag{5.34}$$

$$y^{(4)} = f_s + h_s\rho,$$

$$\varphi(\xi) = f_s = \frac{g}{T_4}\left[\frac{(2mx_1x_2 + T_2)\,x_4 + \left(mgx_1 + \frac{L}{2}Mg\right) cos^2 x_3 + x_4{}^2 \sin x_3}{mx_1^2 + T_1}\right],$$

$$\gamma(\xi) = h_s = \frac{-gcos\ x_3}{T_4(mx_1^2 + T_1)}$$

Now, writing this in the controllable canonical form (phase variable form), one may have

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \vdots \\ \dot{\xi}_4 = \varphi\left(\hat{\xi}\right) + \gamma\left(\hat{\xi}\right)\rho + \gamma\left(\hat{\xi}\right)\Delta G_m\left(\hat{\xi}, \rho, t\right) \end{cases} \tag{5.35}$$

where

$$y^{(i-1)} = \xi_{i=1,2,3,4}, \tag{5.36}$$

5.3.2.1 Sliding Mode Control (SMC)

Here we discuss conventional SMC on the ball and beam system. Sliding manifold defined in (5.6), is considered for $n = 4$ in following equation:

$$\sigma(\xi) = c_1\xi_1 + c_2\xi_2 + c_3\xi_3 + \xi_4 \tag{5.37}$$

Computing the time derivative of (5.37) along (5.35) one may have

$$\dot{\sigma}(\xi) = c_1\xi_2 + c_2\xi_3 + c_3\xi_4 + \dot{\xi}_4 \tag{5.38}$$

Further, after substituting ($\dot{\xi}_4$), ones get

$$\dot{\sigma}(\xi) = c_1\xi_2 + c_2\xi_3 + c_3\xi_4 + \varphi(\hat{\xi}) + \gamma(\hat{\xi})\rho \tag{5.39}$$

Following the procedure laid in Section 5.2.1, the complete control structure becomes:

$$\rho = \frac{1}{h_s}\left[-c_1\xi_2 - c_2\xi_3 - c_3\xi_4 - \varphi(\hat{\xi})\right] - K sign(\sigma) \tag{5.40}$$

where ρ_{eq} and ρ_{sw} are written as (5.41) and (5.42), respectively.

$$\rho_{eq} = \frac{1}{h_s}\left[-c_1\xi_2 - c_2\xi_3 - c_3\xi_4 - \varphi(\hat{\xi})\right] \tag{5.41}$$

$$\rho_{sw} = -K sign(\sigma) \tag{5.42}$$

As the control objective is to perform the reference tracking here, therefore, the sliding manifold and the controller will appear as follows:

$$\sigma(\xi) = c_1(\xi_1 - r_d) + \sum_{i=2}^{4} c_i\xi_i \tag{5.43}$$

where r_d is the desired reference with $\dot{r_d}, \ddot{r_d}, \dddot{r_d}$ are bounded.

5.3.2.2 Second Order Sliding Mode Control (SOSMC)

By following the procedure defined in Section 5.2.2, the sliding manifold σ for super twisting sliding mode control remains the same as defined in (5.37) for $n = 4$. Since reference tracking is objective, therefore, the sliding manifold appears as follows:

$$\sigma(\xi) = c_1(\xi_1 - r_d) + c_2\xi_2 + c_3\xi_3 + \xi_4 \tag{5.44}$$

Moreover, the final structure of the control input is calculated as follows:

$$\rho = -k_1 sign\,(\sigma)\,|\sigma|^{\frac{1}{2}} - k_2\sigma - \int (-k_3 sign\,(\sigma) - k_4\sigma) \tag{5.45}$$

5.3.2.3 Integral Sliding Mode Control (ISMC)

In case of ISMC, the integral manifold defined in (5.23) can be defined as follows for the ball and beam system.

$$\sigma(\xi) = c_1(\xi_1 - r_d) + \sum_{i=2}^{4} c_i \xi_i + z \tag{5.46}$$

The dynamics of the integral term were calculated to be as follows

$$\dot{z} = -c_1\xi_2 + c_2\xi_3 + c_3\xi_4 - \varphi(\hat{\xi}) - \gamma(\hat{\xi})\rho_0 \tag{5.47}$$

The final expression of the controller takes the form:

$$\rho = -k_1(\xi_1 - r_d) - k_2\xi_2 - k_3\xi_3 - k_4\xi_4 + \frac{1}{\gamma(\xi)}\left(-\varphi\left(\hat{\xi}\right) - (\gamma\left(\hat{\xi}\right) - 1)\rho_0 - Ksign(\sigma)\right) \tag{5.48}$$

Note that the higher derivatives $\dot{r_d}, \ddot{r_d}, \dddot{r_d}$ of the reference trajectory were assumed to bounded.

5.3.2.4 Fast Terminal Sliding Mode Control (FTSMC)

In case of FTSMC for the considered ball and beam system with $n = 4$ the fast terminal manifold will be defined as follows:

$$\sigma(S) = \dot{S}(\xi) + \alpha_1 S(\xi) + \beta_1 (S(\xi))^{\frac{p_1}{q_1}},$$

Where $S(\xi) = c_1(\xi_1 - r_d) + c_2\xi_2 + \xi_3$ and p_1 and q_1 are positive odd integers. The final mathematical structure of the applied controller was selected as follows:

$$\rho = (h_s)^{-1} \begin{pmatrix} -f_s - c_2\xi_4 - c_1\xi_3 - \alpha\,(c_1\xi_2 + c_2\xi_3 + \xi_4) \\ -\frac{p_1}{q_1}\beta_1(c_1(\xi_1 - r_d) + c_2\xi_2 + \xi_3(\xi))^{\frac{p_1}{q_1}-1} \\ (c_1(\xi_2) + c_2\xi_3 + \xi_4) - Ksign(\sigma) \end{pmatrix} \tag{5.49}$$

5.4 Simulation Results

In this study, the ball and beam system defined in (5.32) is operated under the action of the control laws (5.40), (5.45), (5.48) and (5.49). The gains used in

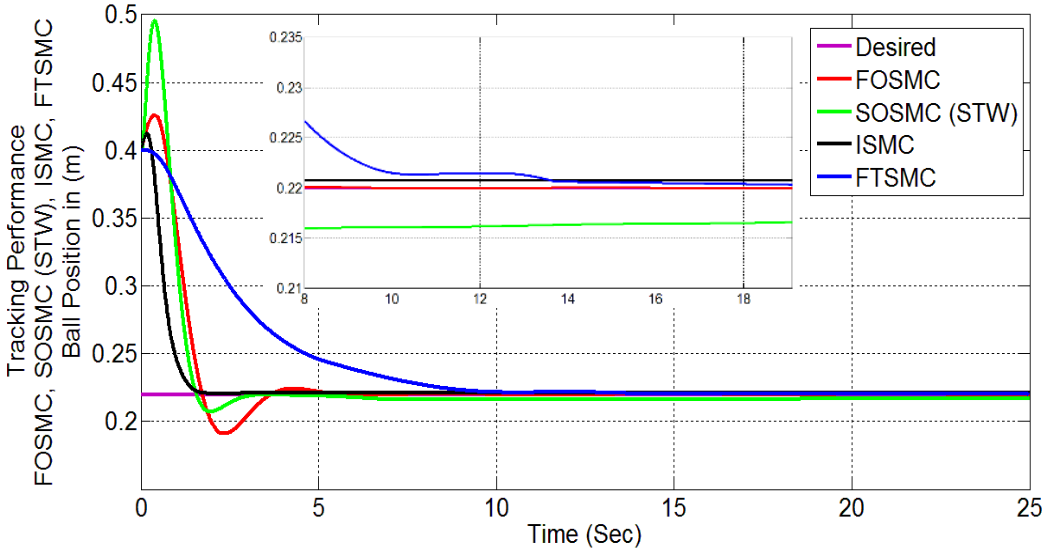

FIGURE 5.1: Output tracking performance of FOSMC, SOSMC(STW), ISMC and FTSMC, $r_d = 22cm$.

controller during simulation are reported in Table 5.1. The computer simulation of the overall closed-loop system is carried by considering the reference tracking to be a fixed point $r_d\,(t)$ and the initial condition of the system was set to be $x_1\,(0) = 0.4, \quad x_2\,(0) = x_3\,(0) = x_4\,(0)$ =0. The reference trajectory was defined to be

$$r_d\,(t) = 22cm \qquad t > 0 \tag{5.50}$$

The output tracking performance of all the four designed controllers is shown in Fig. 5.1. It can be clearly examined that the tracking performance of ISMC is very fast as compared to FOSMC and SOSMC. On the other hand, the performance of FTSMC is slower as compared to the remaining strategies. However, the precision of the FTSMC is very appealing. The zoomed version of reference tracking highlights the convergence precision of all the strategies.

The beam angle stabilization profile for FOSMC, SOSMC (STW), ISMC and FTSMC is displayed in Fig. 5.2. The separated profile of the beam angle stabilization can be seen in Fig. 5.3.

The zoomed version of the angle stabilization shows the steady state error in case of FTSMC as compared to the other stabilization strategies. In the beam

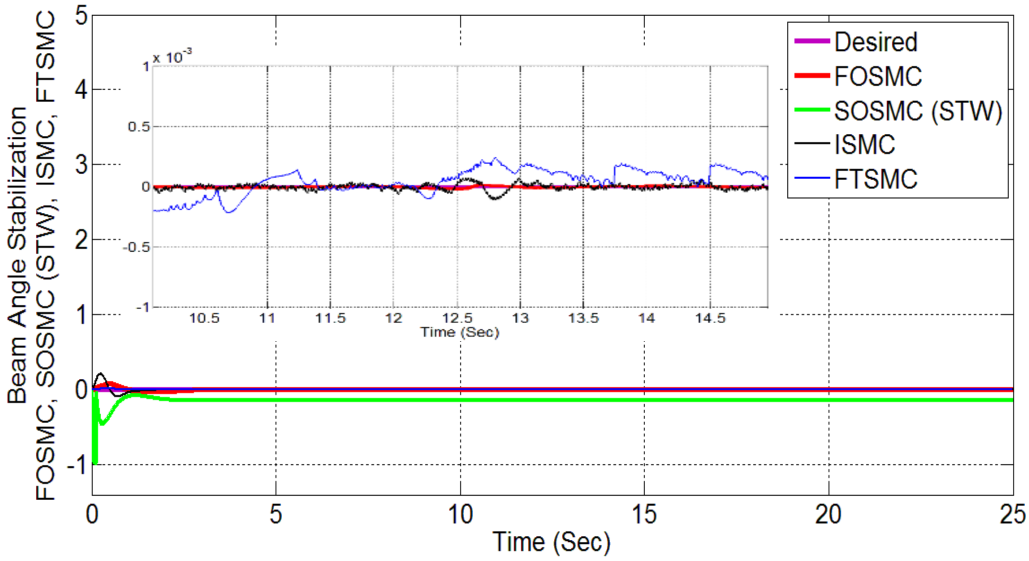

FIGURE 5.2: Beam angle stabilization profile of FOSMC, SOSMC (STW), ISMC and FTSMC, $r_d = 22cm$.

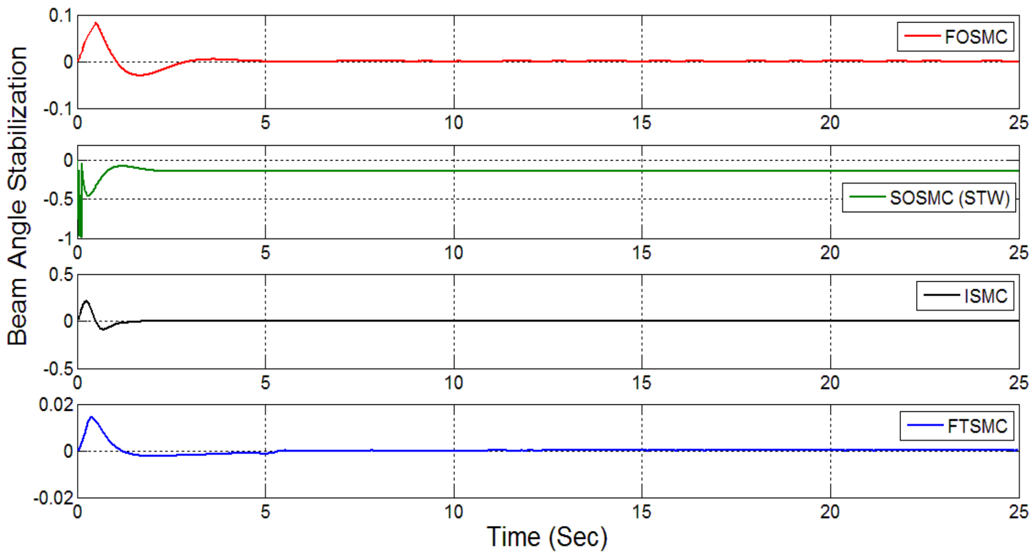

FIGURE 5.3: Separate beam angle stabilization profile of FOSMC, SOSMC(STW), ISMC and FTSMC.

angle stabilization, the FOSMC and ISMC are quite appealing. However, both techniques suffer from chattering which will be discussed in following study.

The sliding manifold comparison of all the techniques is shown in Fig 5.4, and 5.5 with their respective control inputs are displayed in Fig 5.6 and 5.7. In case of manifold convergence, the SOSMC carries substantial marks as compared to its counterparts.

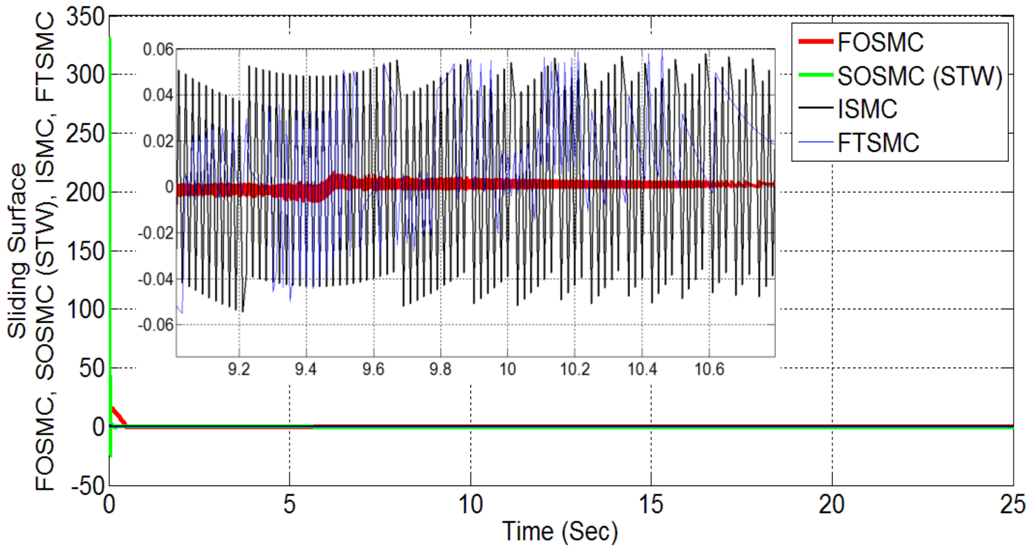

FIGURE 5.4: Sliding manifold convergence profile of FOSMC, SOSMC(STW), ISMC and FTSMC, $r_d = 22cm$.

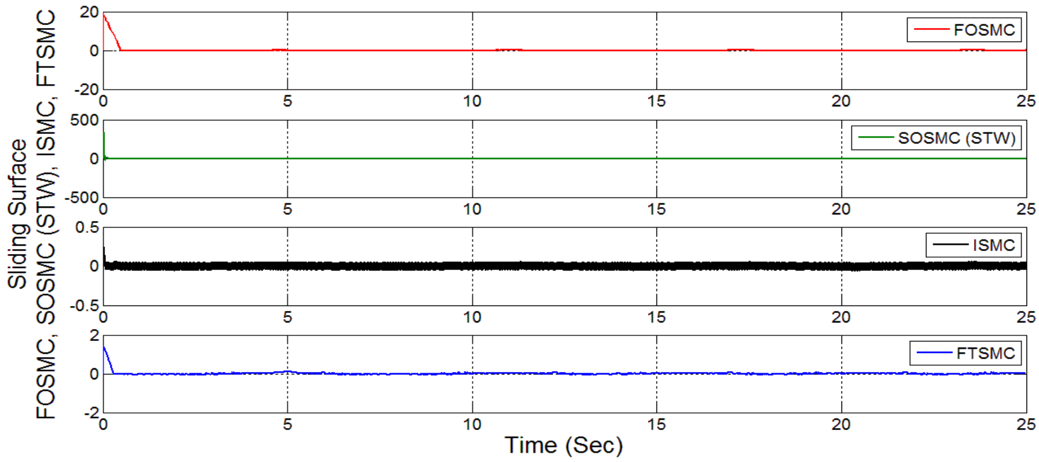

FIGURE 5.5: Separate sliding manifold convergence profile of FOSMC, SOSMC (STW), ISMC and FTSMC.

However, in case of robustness enhancement and reaching phase elimination ISMC is far better than the others. In term of chattering suppression, the FTSMC is better which makes it an appealing candidate in electromechanical systems.

However, the chattering may be reduced in case of ISMC by considering a strong reachability condition. Having chosen strong reachability condition, ISMC will outshine the remaining SMC variants. Note that FOSMC, in this case, suffers from the substantial magnitude of chattering which may cause the system failure. In case of energy consumption, the ISMC, and FTSMC utilize low energy as

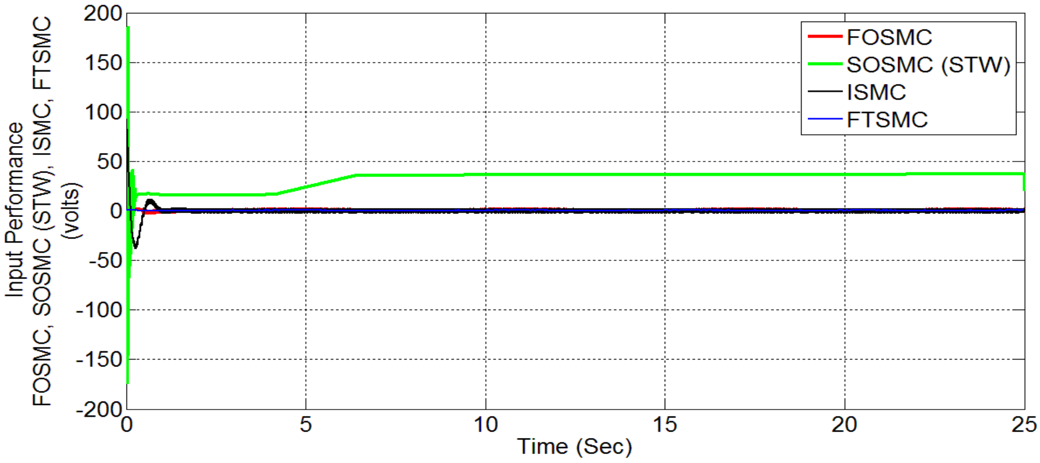

FIGURE 5.6: Control Input profile of FOSMC, SOSMC(STW), ISMC and FTSMC for reference tracking.

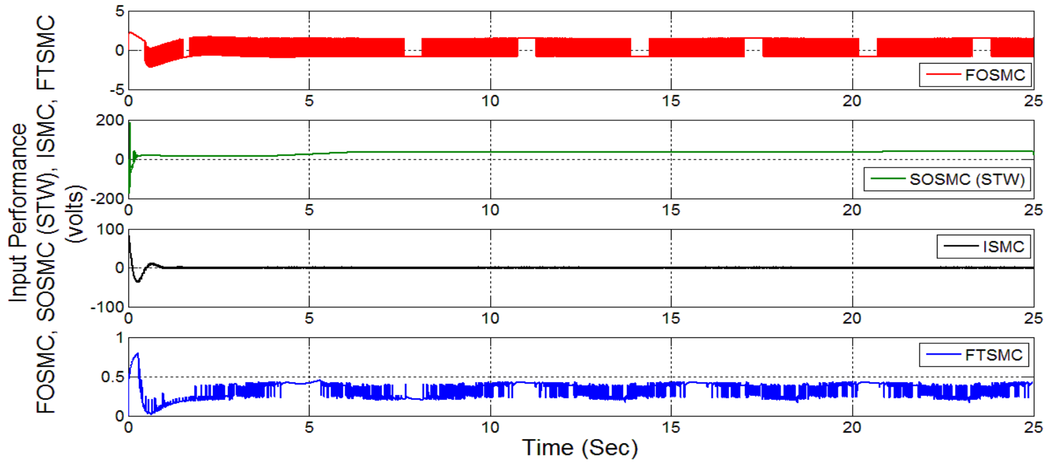

FIGURE 5.7: Separate control input profile of FOSMC, SOSMC(STW), ISMC and FTSMC.

compared to FOSMC and SOSMC. Having analyzed, in our views, the ISMC becomes an appealing candidate to be employed to electromechanical systems.

5.5 Experimental Results

In this study, the core objective is to keep the ball on a beam at the desired position $r_d(t)$. The control algorithms designed in the Section 5.2, are implemented on the actual ball on a beam system. This system is manufactured by Googoltech GBB1004 with an intelligent IPM100 servo drive and an electronic

TABLE 5.1: Parameters values used in the tracking for FOSMC, SOSMC(STW), ISMC and FTSMC.

FOSMC								
Constants	$\mathbf{C_1}$	$\mathbf{C_2}$	$\mathbf{C_3}$	$\mathbf{K_1}$	$\mathbf{K_2}$	$\mathbf{K_3}$	$\mathbf{K_4}$	**K**
In Simulation	80	48	24	-	-	-	-	50
In Practical Implementation	9	4	1	-	-	-	-	1
SOSMC(STW)								
Constants	$\mathbf{C_1}$	$\mathbf{C_2}$	$\mathbf{C_3}$	$\mathbf{K_1}$	$\mathbf{K_2}$	$\mathbf{K_3}$	$\mathbf{K_4}$	**K**
In Simulation	80	48	14	37	0	2	0	25
In Practical Implementation	5	29	5	0.5	0	0.1	0	5
ISMC								
Constants	$\mathbf{C_1}$	$\mathbf{C_2}$	$\mathbf{C_3}$	$\mathbf{K_1}$	$\mathbf{K_2}$	$\mathbf{K_3}$	$\mathbf{K_4}$	**K**
In Simulation	1.2	1.2	0.11	402.98	250.18	60	4.1	5
In Practical Implementation	8	5	1	3	15	3	1	5
FTSMC								
Constants	$\mathbf{C_1}$	$\mathbf{C_2}$	$\mathbf{C_3}$	$\boldsymbol{\alpha_1}$	$\boldsymbol{\beta_1}$	$\mathbf{p_1}$	$\mathbf{q_1}$	**K**
In Simulation	18	10	0.09	0.45	0.01	3	9	1
In Practical Implementation	100	70	10	2	0.11	1	9	12

control box which supports the MATLAB 7.12/Simulink 7.7 environment. Figure 5.8 shows the experimental setup. The other typical parameters of GBB1004 is same as considered/described in Section 4.5.1. Note that the control accuracy of this equipment lies within the range of $\pm 1mm$. During the practical implementation, the sampling time was chosen to be $2ms$. The gains of the controller used during the implementation (experimentation) are reported in Table 5.1. The desired reference point on the beam was 22 cm.

Remark 5.6

In the experimental study, the translational position of the ball on a beam and the angular position of the driving motor are available. The respective velocities are calculated via the built-in velocity estimator. However, one may use reduced order observer (see for instance [7]) for velocity estimation.

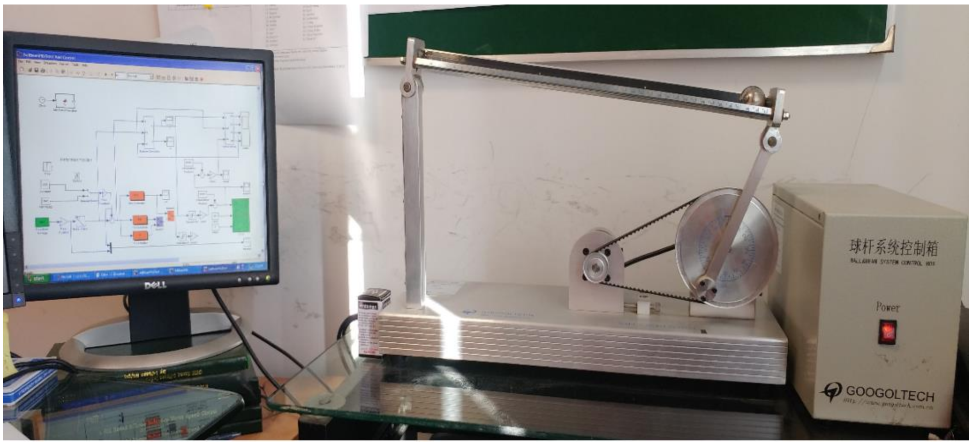

FIGURE 5.8: Experimental Setup of the Ball and Beam equipped via Googoltech GBB1004.

All the four techniques were implemented on the actual system, and their tracking performances are displayed in Fig. 5.9 with zoomed results shown in Fig. 5.10. These results follow lies within the vicinity of $0.22m$ which follow the physical limitation of the system. It is clear from Fig. 5.9 that the performance of the FTSMC and ISMC shows slower convergence to the reference point. However, the steady-state error of these two techniques is quite smaller than the FOSMC and SOSMC. The precision of FTSMC carries comparatively high marks as compared to ISMC. It is noticeable that the ISMC result observes small oscillations and shows a very stable behavior in the vicinity of the reference point. The results of SOSMC, in this case, is quite impressive and quite acceptable as compared to FTSMC and ISMC, but it loses precision. The results of FOSMC are not acceptable because it exhibits oscillatory behavior along with low precision.

The comparative and separate beam angle stabilization profile for FOSMC, SOSMC (STW), ISMC and FTSMC are displayed in Figs. 5.11 and 5.12, respectively. It is worthy to notice that high chattering appears in the beam angle while implementing FOSMC. Comparatively, little chattering appears in case of ISMC and SOSMC. However, the beam angle of FTSMC exhibits with continuous chattering with sufficient small amplitude. Although it is tolerable for this system yet, it is quite dangerous in practical systems when they are supposed to be operated for a long time. In the author views, the ISMC and SOSMC are quite appealing in this case because once the angle is stabilized the system observes no chattering and in

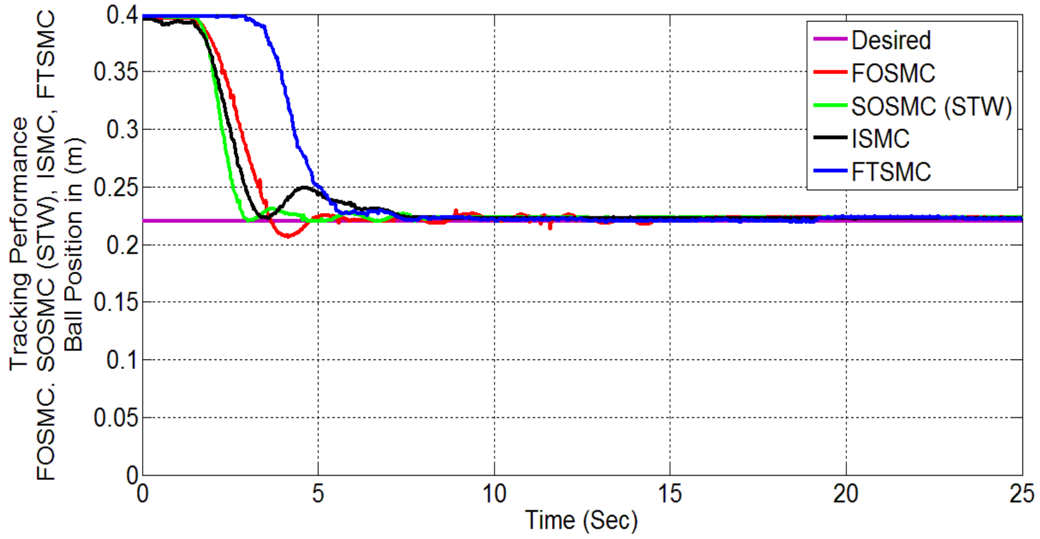

FIGURE 5.9: Output tracking performance of FOSMC, SOSMC(STW), ISMC and FTSMC, $r_d = 22cm$.

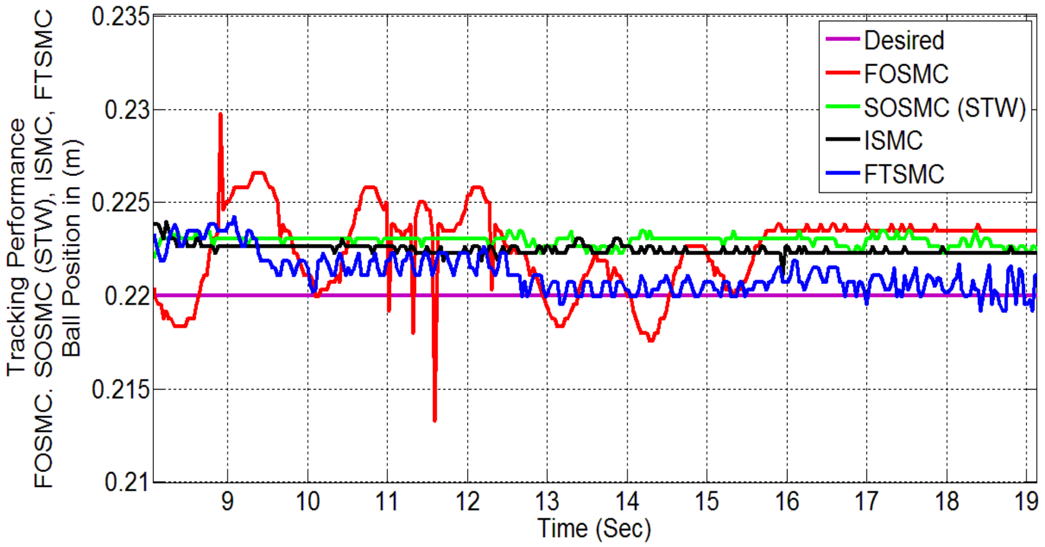

FIGURE 5.10: Zoom profile of output tracking performance of FOSMC, SOSMC(STW), ISMC and FTSMC.

this case, it may not be that harmful to the system health. We summarize this behavior as follows:

FOSMC:	have maximum (high amplitude) chattering,
SOSMC(STW):	have moderate chattering,
ISMC:	have (more than STW, less than FOSMC),
FTSM:	have minimal chattering but continuous.

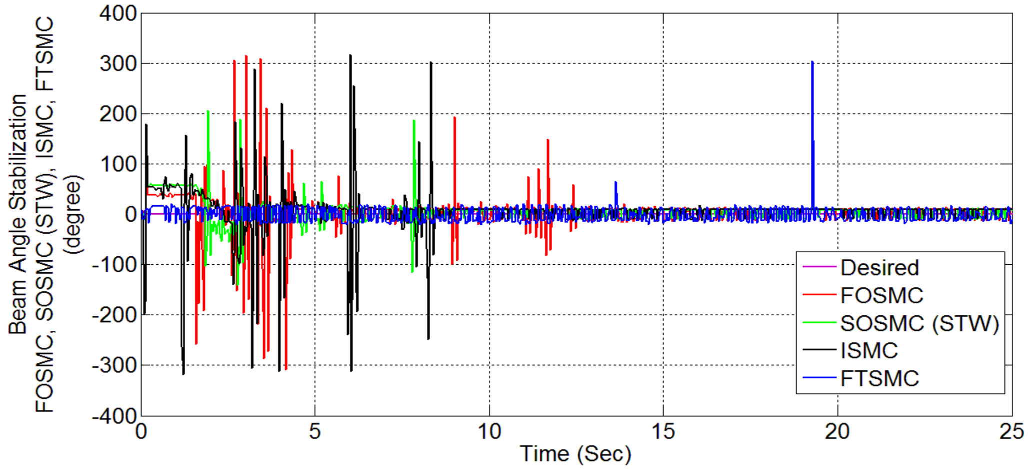

FIGURE 5.11: Beam angle stabilization profile of FOSMC, SOSMC(STW), ISMC and FTSMC, $r_d = 22cm$.

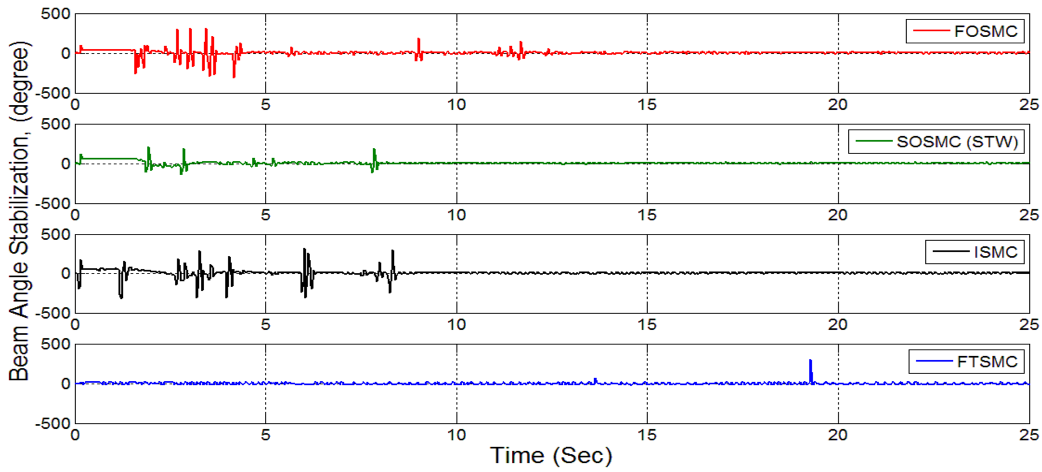

FIGURE 5.12: Separate beam angle stabilization profile of FOSMC, SOSMC(STW), ISMC and FTSMC.

The sliding manifold convergence comparison, as well as separate profiles of all the techniques, are shown in Figs. 5.13 and 5.14. The control efforts of these algorithms are also displayed in Figs. 5.15 and 5.16. It is clear that all the four controllers have stable sliding manifolds converge toward the origin. Figure 5.14 experimentally verified the sliding mode enforcements.

The manifold of the ISMC remains almost at zero which confirms the results achieved in the simulations (see Fig. 5.4). The manifold of the FTSMC observes some massive peaks which may threaten the health of the system. The SOSMC and FOSMC manifold convergence is somewhat interesting.

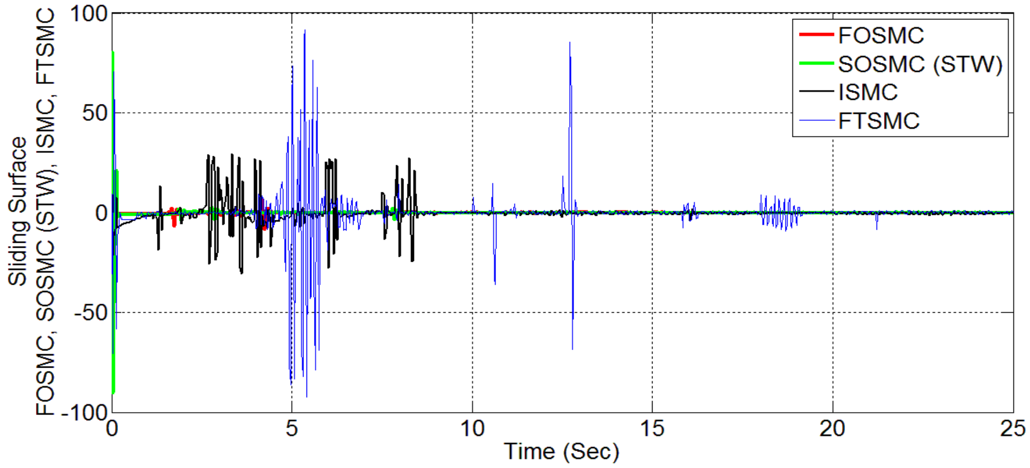

FIGURE 5.13: Sliding manifold convergence profile of FOSMC, SOSMC(STW), ISMC and FTSMC, $r_d = 22cm$.

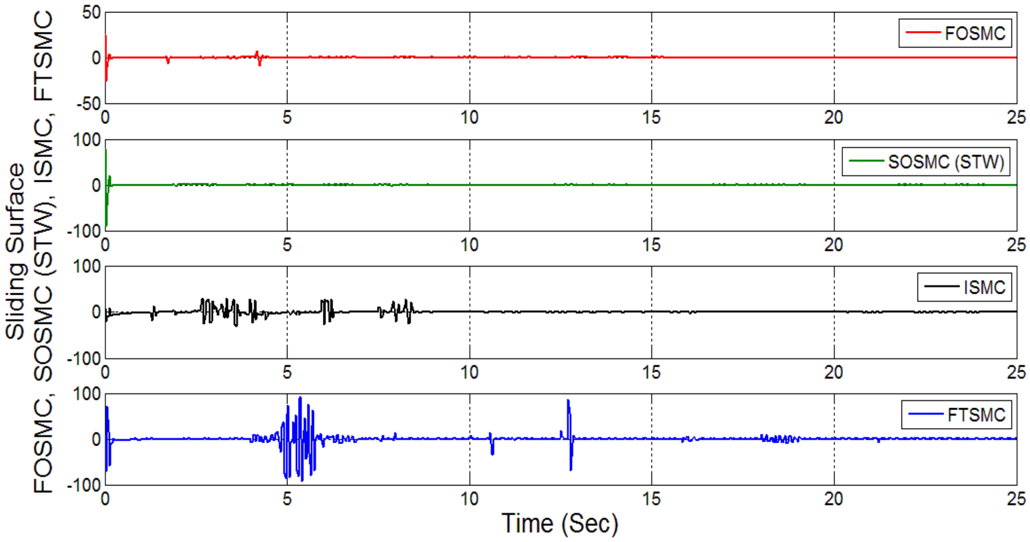

FIGURE 5.14: Separate sliding manifold convergence profile of FOSMC, SOSMC(STW), ISMC and FTSMC.

On the other hand, the control efforts of FTSMC is quite higher even when the system is in sliding mode. This behavior reduces the applicability of FTSMC. While looking at FOSMC control efforts and SOSMC efforts, they fascinate the control designer to use these two strategies. However, the tracking behaviors of both these techniques disappoints as compared to ISMC. In term of robustness, ISMC remains more robust than the other the other three strategies because of the reaching phase elimination.

Further, on Googoltech GBB1004 platform, the utilized energy comparison among

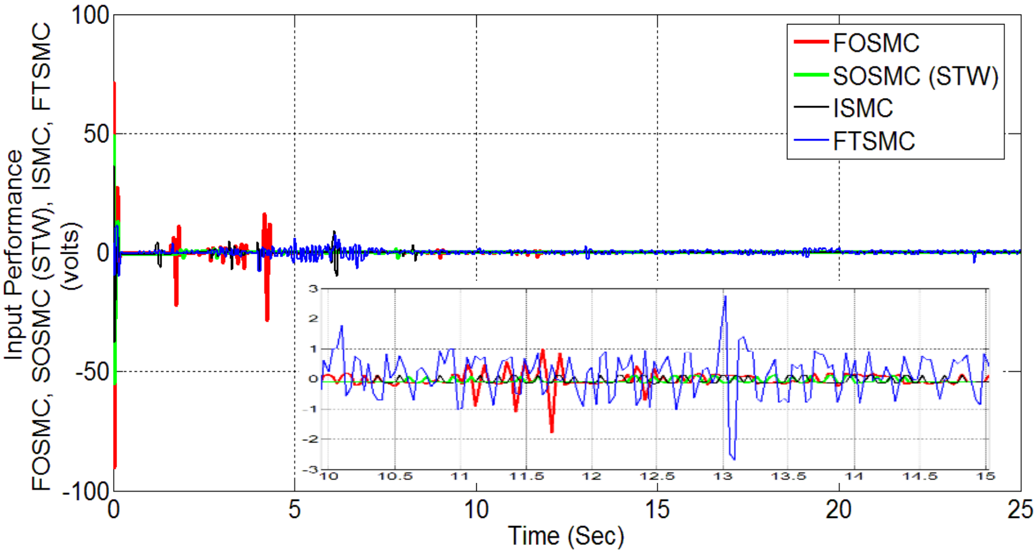

FIGURE 5.15: Control Input profile of FOSMC, SOSMC(STW), ISMC and FTSMC for reference tracking.

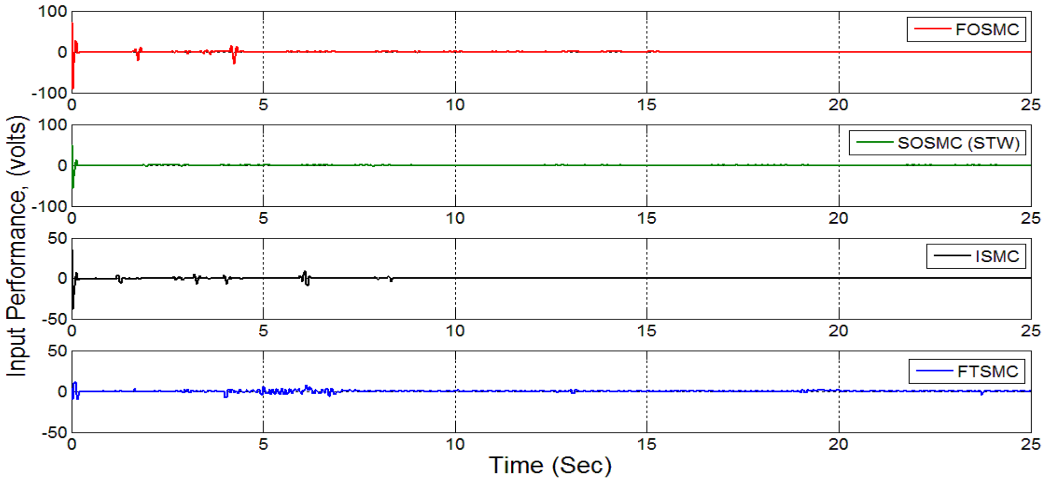

FIGURE 5.16: Separate control input profile of FOSMC, SOSMC(STW), ISMC and FTSMC.

TABLE 5.2: Comparison of energy criteria ($\times 10^4$ J).

FOSMC	**SOSMC(STW)**	**ISMC**	**FTSMC**
4.1072	3.2961	2.3461	2.5712

FOSMC, SOSMC(STW), ISMC and FTSMC is performed using the criteria presented in [1]:

$$J = \int_0^{15} u^2 dt \tag{5.51}$$

which is proportional to the energy delivered to the system. While comparing the energy utilization of these four algorithms, it comes out that ISMC utilizes minimum energy. The second minimum energy is utilized by the FTSMC. The third position is occupied by the SOSMC (see for details, Table 5.2).

5.6 Performance Analysis

The overall performance analysis is summarized in the form of Table 5.3, based on different features in the experimental and simulation results. Having analyzed, it was decided that ISMC carries substantial marks in case of robustness and fast convergence with considerable suppression in chattering. However, in case of precision and chattering suppression, the FTSMC can be preferred. The named FOSMC exhibits high-frequency vibrations with the considerable magnitude and with significant steady-state error whereas the SOSMC suffers from robustness issues in the reaching phase with serious steady-state error. According to the attributes presented in Table 5.3, it can be claimed that ISMC proves itself to be an appealing control protocol for the class of underactuated systems.

5.7 Summary

A comprehensive comparative simulation and experimental study of the FOSMC, ISMC, SOSMC, and FTSMC has been carried out in this work. The experimental step which was considered in this work was the ball and beam system. Before the design, the system was transformed to a controllable canonical form, and then the control inputs were designed via the aforesaid strategies. The experimental and simulation study was carried out in the MATLAB environment. Comprehensive comparative analysis proves ISMC is best suitable for robustness and fast convergence with suppressed chattering. However, when high precision is required FTSMC may be preferred. The benefit of this study is to analyze the appealing attitude of FTSMC and ISMC in electromechanical systems.

Table 5.3: Comparative analysis FOSMC, SOSMC(STW), ISMC and FTSMC.

Attributes	**FOSMC**	**SOSMC(STW)**	**ISMC**	**FTSMC**
Tracking Control	Slow (Not precise)	Slow (Not precise)	Very fast (Precise)	Slow (Highly precise)
Settling Time	Low	Low	Very low	High
Overshoot	High	Very high	Minimal	No overshoot
Chattering Analysis	Severe chattering	Low chattering	Minimal chattering	Moderate chattering
Sliding Surface Convergence	To origin, with chattering of medium magnitude amplitude	Remains at the origin with small magnitude oscillations in the very start	To origin, with moderate chattering amplitude	To origin, with considerable high chattering amplitude
Control Effort	Very high	High	Lowest	Low
Energy Utilization (in terms of Joule $\times 10^4$)	4.1075 (Maximum)	3.2961	2.3461 (Lowest)	2.5712
Computational Complexity	Low	High	High	High

Chapter 6

Conclusion and Future Work

6.1 Conclusion

The underactuated nonlinear systems are always equipped with less number of actuators than the system's outputs. This feature offers certain benefits like reduction in weight and minimum energy usage. Along with certain benefits, there always remain some drawbacks which is needed to be addressed. So far, many control strategies are applied to such systems. Some of them includes passivity approach, back stepping approach, fuzzy control, optimal control, and sliding mode control. In the aforesaid control strategies, sliding mode control acquires so much attention from the researchers due to its robust response in the existence of uncertainties. It has been widely employed to the class of UAS in the last decades. However due to the variations in the dynamic structures of UAS, it is not applied in the general scenario. In this work, main emphasis is the transformation of this class to a more generalized form and the accomplishment of recent techniques. In addition, the high frequency vibrations of the control law in sliding mode is considered to be dangerous. Therefore, an integral sliding mode control approach is proposed for this class. This strategy shows robustness from the very start of the process. This technique along with improved robustness results in chattering reduction. The asymptotic tracking is achieved which is more appealing in real applications. This proposed ISMC is practically applied to a ball on a beam system and quite fruitful implementation is observed.

In addition, comprehensive comparative simulation and experimental study of the FOSMC, ISMC, SOSMC and FTSMC has been carried out. The experimental and simulation study was performed in the MATLAB environment. Having analyzed, it was decided that ISMC carries marks in case of robustness and fast convergence with considerable suppression in chattering. However, in case of precision and chattering suppression the FTSMC can be preferred. The named FOSMC exhibited high frequency vibrations with considerable magnitude and with substantial steady state error whereas the SOSMC suffers from robustness issues in the reaching phase with considerable steady state error. The benefits of this study is to analyze the appealing attributes and characteristics in practical underactuated systems. Now, the future directions are outlined in the subsequent section.

6.2 Future Research Directions

The proposed work can be extended theoretically as well as from application perspectives in following ways.

- ISMC can be utilized in fusion with FTSMC or neural network.
- Adaptive ISMC can be implied instead of conventional ISMC in fusion with fuzzy or neural network control strategy.
- Smooth super twisting control algorithm can be implied for the specific class considering the aim of robust stabilization/tracking.
- Modeling with friction is ignored in the mathematical modeling of underactuated systems, which yet need to be explored further.
- Window of research is also present regarding flat underactuated systems.

- New barriers can be crossed in the dimension of bounded control inputs, to counter the saturated nonlinear state feedback problem.

- By utilizing the Micro Electromechanical Systems (MEMS) technology, there is need to build small test benches for the underactuated systems.

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